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# Plane and Solid
# Geometry

33

# Plane and Solid
# Geometry
## *Suggestive Method*

By

Clinton

George C. Shutts
*Instructor in Mathematics, State Normal School*
*Whitewater, Wisconsin*

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# THE PREFACE.

O 11-13-29 H.C.M.

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THE present book is a revision of the Van Velzer & Shutts Geometry, in the "Suggestive Method," which has been privately printed for some time. The material of the book was used for several years from mimeograph notes with such excellent results that it was thought best to put it in text-book form. After being thoroughly tested in many of the best schools in the country, the work has been again revised in an attempt to make it more suggestive to the teacher and more helpful to the student.

In putting the work into its present form the scientific classification of the subject-matter has been departed from when it was thought that by so doing the work could be better graded to the ability of the average pupil. For this reason the subject of the triangle has been introduced before the relation of lines and angles has been fully discussed.

The treatment of the theory of measurement has been modified to make it more easily understood. A treatment of the application of proportion has been suggested that will make the pupil more independent in his work, and at the same time it has not increased the difficulty of the subject.

The book, as now arranged, is sufficient for all college entrance requirements, yet it can be completed, including all the exercises, by high-school pupils in one school year. For those schools that devote a year and a half to the subject additional work has been placed in the Appendix, to which references are given in the text, so that the various propositions and exercises can be taken up in logical order. A fuller treatment of the theory of limits, which many teachers desire, is also given in the Appendix.

A departure from ordinary methods will be noticed in the treatment of proportion. It has not been thought wise to follow the usual method of limiting the subject to proportions whose terms are pure numbers, nor yet to follow the Euclidian method common in England, which admits of proportions whose terms are concrete magnitudes, but which is so difficult that it can be understood by only the best students. The method in the text will be found to admit of proportions whose terms are concrete magnitudes, yet it will do no violence to the fundamental ideas of Arithmetic regarding operations upon concrete magnitudes. It is believed that the subject of limits is treated in so simple a manner that beginners can grasp it.

An edition, consisting of the theorems, the diagrams, the things given, and the things to prove, *without* suggestions for demonstration, will accompany the book for class-room use. It is hoped that this will be found as valuable in Geometry teaching as "text editions" have been in teaching the classics. This "Class-Room Edition" will save the time of the recitation usually consumed by the pupils in drawing the figures upon the blackboard. The complete edition of the book can be banished from the recitation and the temptation to get assistance from the text eliminated. The blackboards can thus be reserved for original demonstrations of exercises and for suggestive work by the teacher. The Class-Room Edition will lengthen the daily recitation, and make it possible for more work to be done in a year.

The author wishes to thank those who have given the previous edition of the book so kind a reception. It is hoped that the present book will more fully meet the needs of all teachers who wish their pupils to make the largest possible growth in independent thinking in Geometry.

Acknowledgment should be made of the scholarly criticism of Dr. C. A. Van Velzer, *Head of the Department of Mathematics in the University of Wisconsin,* for his helpful services in preparing for publication the manuscript of the first edition of the book.

Thanks are also due for valuable suggestions in reading the proof to Mr. C. E. Bunsa, *Superintendent of Schools, Columbus, Wisconsin;* Mr. R. L. Sandwick, *Principal of the Deerfield Township High School, Highland Park, Illinois;* Miss Genevieve Decker, *Teacher of Mathematics in the High School, Janesville, Wisconsin,* and to Miss Maud Averill, *Teacher of Mathematics in the High School, Whitewater, Wisconsin.*

G. C. S.

*Whitewater, Wis., August 25, 1904.*

# SUGGESTIONS TO TEACHERS.

GEOMETRY is essentially a disciplinary study. The value derived from its study is in proportion to the amount of independent thought expended by the pupil. A text-book in Geometry is in the nature of a "key" to the extent to which the demonstrations are written out for the pupil. That part of the work which a pupil can do for himself should not be done for him. The teacher and text-book should furnish the pupil with data and stimulate thought rather than give him a set form of words which he may repeat verbatim, with or without, the ideas which these words should express.

In this Geometry suggestions arranged in logical order take the place of detailed demonstration. These suggestions are intended to stimulate and direct the thought of the pupil so that he may largely work out his own demonstrations.

Model demonstrations are given of a few propositions to show the student the *form* in which they should be presented. The answers to the suggestions, logically arranged, constitute the demonstration. The suggestions should be studied in the order given, for each suggestion usually depends. upon the preceding one. The answer to a suggestion should consist of a statement of the relations asked for, together with the authority in full for such statement.

To permit the pupil to ignore the authority is to encourage carelessness, slovenliness, and inaccuracy in demonstration. A common error is to apply authority that does not exactly fit the conditions under consideration. The pupil must understand that the authority should, without exception, be a definition, an axiom, or a previously proved proposition. "It seems so," or, "it looks reasonable," or any expression of judgment will not do. The pupil should be encouraged to search out his own authority, even when the authority is quoted for him in the suggestions, and to use the reference simply for verification. A pride in independent work is a most important factor in securing satisfactory results.

In the preparation of the lesson the pupil should write out his demonstration, noting carefully the form of the "models." This will insure correct form and avoid haziness of thought. During the first few weeks this written work, as well as tests taken in the recitation, should be read by the teacher and returned to the pupil for correction.

The exercises, or at least a part of them, should be demonstrated daily along with the propositions as they occur, and not be studied all together at the end of a chapter.

The best results will be obtained by starting slowly, reviewing frequently, and passing over nothing that is not clearly understood. Since each demonstration involves previous propositions and definitions, facility in demonstration can best be secured by committing to memory each theorem, definition and axiom; for that authority cannot be readily recognized and applied which is imperfectly remembered. The demonstrations should not be committed to memory.

The subject of proportion is probably the most difficult part of Geometry. Clearness of thought in the applications of proportion can be obtained only by careful illustration and rigid demonstration in the theory. To teach the theory of proportion by means of numbers, and then to apply the principles developed to geometric magnitudes and numbers indiscriminately without consideration of the limitations of various statements, is not scientific. Note —, page —, should receive careful attention. In deriving the form $A = m B$ from $\frac{A}{B} = m$ the tendency is to claim the multiplication of both members of the equation by B. This is correct if B is a number, but the process is unthinkable if B is a geometric magnitude. $\frac{A}{B}$ means that A is divided or measured by the unit B, hence to say that A contains B, m times, is simply another way of saying that A is m times the unit B, or m B. 12 contains 4 three times $(\frac{12}{4} = 3)$ is another form of expression for 12 is equal to 3 fours $(12 = 3 \times 4)$. The expression $\frac{1 \text{ foot}}{1 \text{ inch}} = 12$, means the same as the expression 1 foot is equal to 12 inches. In this connection see Art. —.

# PLANE GEOMETRY.

## CHAPTER I.

## RECTILINEAR FIGURES.

### Definitions.

**1.** The block represented in the accompanying figure occupies a limited portion of space. If we imagine the block to be removed, its form or shape can still be retained in the mind. This is true of any object or body.

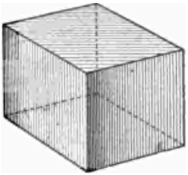


FIG. 1.

The space conceived to be occupied by an object or body as distinguished from the substance of which it is made, is a **geometrical solid.** The matter or substance of which a body or object is composed is a physical solid. Hence a geometrical solid is the shape or form of a physical solid, or some form or figure conceived by the mind.

A **geometrical solid** is a limited portion of space, and has length, breadth, and thickness.

The term solid will be used hereafter to signify a geometrical solid.

**2.** When space is divided into distinct portions or geometrical solids, the boundaries of these portions or solids are **surfaces.** Distinct portions of the bounding surface are **faces.**

**Surface** has length and breadth, but no thickness.

3. When a surface is divided into distinct portions, the boundaries of these portions are **lines.** In the solid, represented in Fig. 1, the edges, or boundaries of the faces, are lines. These lines, being the intersection of faces which have no thickness, can themselves have neither breadth nor thickness.

A **line** has length, but neither breadth nor thickness.

4. When a line is divided into distinct portions, the limits of these portions are **points.** In the solid, represented in Fig. 1, the corners, or limits of the edges, are points. These points, being the intersections of lines which have neither breadth nor thickness, can themselves have neither length, breadth, nor thickness.

A **point** has position, but neither length, breadth, nor thickness.

5. A surface can be conceived of apart from a solid, a line apart from a surface, and a point apart from a line. If a point is conceived to move, the path in which it moves is a line. Hence a line is the path, or **locus,** of a moving point.

A line can be thought of as generated by a **point in motion;** surface can be thought of as generated by a **line in motion;** a solid, as generated by a **surface in motion.**

6. A **geometrical figure** is a combination of points, lines, surfaces, or solids.

Geometrical figures are ideal, that is, they are mental conceptions, but they can be represented to the eye only by material substances. For instance, a line can be represented by a mark made by a pencil or crayon; a solid can be represented by a drawing, by a block of wood, or by some other material of any given shape.

To avoid multiplying words, the material representation of geometrical figures will be generally referred to as standing for the mental conceptions themselves. The pupil will be able to tell by the context whether the word "figure" refers to a geometrical figure or to the material representation of a figure.

7. A **straight line** is a line such that any part of it, however placed, lies wholly in any other part if its extremities lie in that part. Let *O B*, which is any part of *A B*, be placed upon some other part in any way, except that *O* and *B* shall lie upon that part, for instance with *O* at *M* and *B* at *N*. If *O B* exactly coincides with *M N*, *A B* is a straight line. Illustrate by lines represented by wood, paper, or other material.

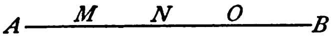


FIG. 2.

A line is **read** by naming letters placed at its extremities, as line *A B* in Fig. 3; or by naming a single letter placed upon it, as line *O* in Fig. 9.

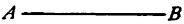


FIG. 3.

8. A **broken line** is a line made up of a succession of different straight lines, as *A B C D E*, in Fig. 4.

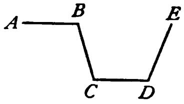


FIG. 4.

9. A **curved line**, or a **curve**, is a line no portion of which is straight, as *C D*, in Fig. 5.

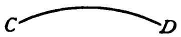


FIG. 5.

10. A **plane surface**, or a **plane**, is a surface such that if any two of its points be joined by a straight line the line lies wholly in the plane surface. If a carpenter wishes to determine whether or not the surface of a board

is a plane, he tries to place a straight edge so that at least two of its points touch the surface. If his straight edge lies continuously in the surface, it is a plane surface.

11. A **plane figure** is a figure that lies wholly in the same plane.

12. A plane figure, in which the lines are all straight lines, is a **rectilinear figure.**

13. **Magnitudes** are figures considered only with reference to extent.

14. **Geometry** is the science that treats of points, lines, surfaces, and solids, and is concerned with the construction and measurement of geometrical figures.

15. **Plane geometry** treats of plane figures.

16. **Solid geometry** treats of figures which are not wholly in the same plane.

### Angles.

17. When two straight lines meet or intersect, they contain, or make with each other, **an angle.**

The two lines are the **sides** or **arms** of the angle, and the point of meeting is its **vertex.**

(*a*) An angle can be **read** by naming the letter at the vertex of the angle between the letters upon the sides of the angle, as angle *A B C* or angle *C B A* in Fig. 6. When there is only one angle at a given vertex, it is sufficient to read the letter at the vertex, as angle *B*. When two or more angles have a common vertex, letters or figures are frequently placed near the vertex between the sides of the angles to designate the angles, as *m* and *n* in Fig 7. For example,

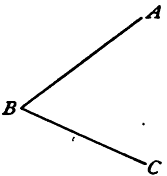


FIG. 6.

we say angle *m* instead of angle *C B A*, and angle *n* instead of angle *C B D*.

(*b*) When a line, coincident with one side of an angle, revolves about the vertex remaining always in the same plane until it arrives at the position of the other side of the angle, the line **turns through the angle,** and the greater the amount of turning the greater the angle.

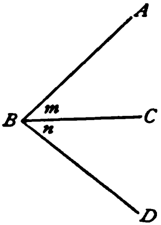


FIG. 7.

(*c*) Hence the **magnitude** of an angle depends upon the amount of revolution necessary to turn a line through the angle. The length of the sides of the angle bears no relation to the size of the angle. The magnitude of an angle may be made clear by means of a pair of dividers, the legs of the dividers representing the sides of the angle and the hinge, the vertex. If the dividers are opened a given amount, a certain angle is represented; if from that position they are closed more or less, a smaller angle is represented; if they are opened farther, a larger angle is represented.

(*d*) A line can turn through an angle in two directions, hence there are two angles which have the same sides and the same vertex.

For example, the side *O B*, in Fig. 8, can be made to turn through the angle *m* by a motion opposite to that of the hands of a watch, to the position *O A*, or it can be made to turn through the angle *n* by a motion like that of the hands of a watch, to the same position *O A*.

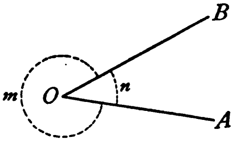


FIG. 8.

Two angles which have the same

sides and the same vertex are **conjugate angles.** When an angle is referred to, the smaller of the two conjugate angles is always meant, unless the other is specifically mentioned.

(*e*) The **direction of a line** is its position as determined by the angle it makes with a given line upon a given side of it. For instance, the direction northeast means a line which makes an angle of forty-five degrees with a north and south line on the east side of it.

A surveyor indicates a particular direction when he says "south 17° 20′ east." He means the line which makes an angle of 17° 20′ with the north and south line on the east side of it.

**18.** When the two sides of two conjugate angles lie in the same straight line, each conjugate angle is a **straight angle;** for example, the angle *A O B* in Fig. 9 is a straight angle.

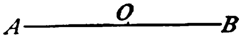


FIG. 9.

The two sides of a straight angle form a straight line.

**19.** Two angles which have a common vertex and one common side, and are on opposite sides of this common side, are **adjacent angles.** In Fig. 10, angles *1* and *2* are adjacent angles.

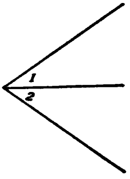


FIG. 10.

**20.** A **right angle** is an angle made by two straight lines which meet so that the adjacent angles formed are equal. In Fig. 10, if *A O* and *B C* meet so that the adjacent angles *1* and *2* are equal, angle *1* is a right angle; angle *2* is also a right angle.

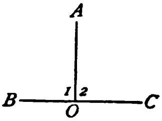


FIG. 11.

**21.** An **acute angle** is an angle that is less than a right angle. In Fig. 12, *B O C* is an acute angle.

**22.** An **obtuse angle** is an angle that is greater than a right angle. In the same figure, *A O B* is an obtuse angle.

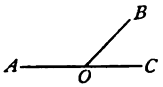


FIG. 12.

Acute and obtuse angles are **oblique angles.**

**23.** A **perpendicular line,** or a **perpendicular,** is one that makes right angles with another line. In Fig. 11, *A O* is perpendicular to *B C*, and *B C* is perpendicular to *A O*.

**24.** A line is **oblique** to another line when it makes oblique angles with that line. The two lines are sometimes called oblique lines. In the Fig. 12, *B O* is oblique to *A C*, and *A C* is oblique to *B O*.

**25.** When two lines intersect, the opposite angles are **vertical angles.** The angles *m* and *n* are vertical angles; angles *p* and *o* are also vertical angles.

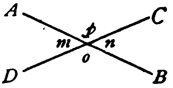


FIG. 13.

**26.** Two angles are **complementary** if their sum equals one right angle. The angles are then **complements** of each other.

If in Fig. 14 *M B* is perpendicular to *A D*, angles *1* and *2* are complements of each other. If angles *A* and *B*, Fig. 15, are together equal to one right angle, they are complements of each other.

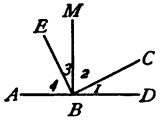


FIG. 14.

**27.** Two adjacent angles that are complements of each other are **complementary adjacent angles,** as angles *1* and *2*, Fig. 14.

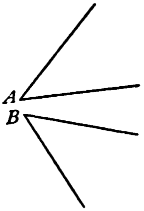


FIG. 15.

**28.** Two angles are **supplementary** if their sum equals two right angles. The angles are then **supplements** of each other. Angle *1* and angle *A B C*, in Fig 14, and the angles *m* and *p*, in Fig. 13, are supplementary angles. If in Fig. 15 the sum of angles *A* and *B* is equal to two right angles, *A* and *B* are supplementary angles.

29. Two adjacent angles which are supplements of each other are **supplementary adjacent angles.**

## Logical Terms.

30. A **theorem** is a truth which requires demonstration. For example: *If two straight lines intersect each other, the vertical angles are equal.*

31. The statement of a theorem is its **enunciation,** or the **general enunciation.**

When a drawing is made to illustrate a theorem, the description of the drawing is the **special enunciation.**

32. A theorem consists of two parts, the **hypothesis** and the **conclusion.** The conditional part of a theorem is the **hypothesis.** For example, in the above theorem the hypothesis is: "*If two straight lines intersect each other.*" The hypothesis is sometimes called the **premises.**

The truth depending upon, or following from, the hypothesis is the **conclusion.**

In the theorem stated in article 30, the truth, "*the vertical angles are equal,*" depends upon the hypothesis, "*If two straight lines intersect each other,*" and is therefore the conclusion.

33. The **demonstration,** or **proof,** of a theorem is the course of reasoning by which the truth of the theorem is established.

34. A **problem** is a question proposed for solution, or the statement of certain relations which are to be pro-

duced. For instance: *To construct a line perpendicular to a given line at a given point.*

35. A **proposition** is a general term for a theorem or problem.

36. A **corollary** is a proposition easily deduced from the proposition to which it is attached, with the aid, if necessary, of one or more previous propositions.

37. A **scholium** is a remark upon one or more propositions with respect to their applications, limitations, or connections.

38. An **axiom** is a theorem which, from its simplicity, must be admitted without demonstration: as, *The whole of anything is equal to the sum of all its parts.*

39. A **postulate** is a problem whose solution, on account of its simplicity, is admitted to be possible: as, *Let it be granted that a straight line can be drawn between two points.*

### Axioms.

40. *1.* Things which are equal to the same thing or equal things, are equal to each other.

*2.* If equals are added to equals the sums are equal.

*3.* If equals are subtracted from equals the differences are equal.

*4.* If equals are multiplied by the same number or by equals the products are equal.

Corollary.—Doubles of equals are equal.

*5.* If equals are divided by the same number or by equals the quotients are equal.

Corollary.—Halves of equals are equal.

*6.* If equals are added to unequals the sums are unequal, and that sum is the greater which is obtained by adding to the greater magnitude.

Corollary.—If unequals are multiplied by equals the products are unequal, and that product is the greater which is obtained by multiplying the greater magnitude.

*7.* If equals are subtracted from unequals the differences are unequal, and that difference is the greater which is obtained by subtracting from the greater magnitude.

Corollary.—If unequals are divided by equals, the quotients are unequal, and that quotient is the greater which is obtained by dividing the greater magnitude.

*8.* If unequals are subtracted from equals the differences are unequal, and that difference is less which is obtained by subtracting the greater magnitude.

*9.* The whole is greater than any of its parts.

*10.* The whole is equal to the sum of all its parts.

*11.* A straight line is the shortest distance between two points.

*12.* If two straight lines have two points in common they are one and the same straight line.

Corollary 1.—Two straight lines can intersect in but one point.

Corollary 2.—But one straight line can be drawn between two points.

*13.* Magnitudes which coincide are equal in all respects, and conversely, magnitudes which are equal may be made to coincide.

*14.* Every magnitude has two halves or is equal to two halves of itself.

Corollary.—Every magnitude, however small, may be divided into two or more parts.

*15.* If any magnitude, however small, is added to one of two equal magnitudes, and the same subtracted from the other, the results are unequal.

*16.* There is only one shortest line between two points, between a point and a line and between two parallel lines.

*17.* A magnitude which is less or greater than one of two equal magnitudes is less or greater, respectively, than the other.

*18.* Only one line can be drawn through a given point parallel to a given line.

Postulate 1.—A line can be revolved about a given point until it embraces another point or takes the direction of a line drawn through the given point.

Postulate 2.—A figure can be thought of as being changed in position without making any change in the relation of its parts.

### Symbols and Abbreviations.

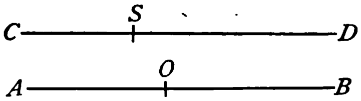
**41.** ∠ —*angle*
∠s—*angles*
⊥ —*perpendicular*
⊥s—*perpendiculars*
∥ —*parallel*
∥ s—*parallels*
△ —*triangle*
△s—*triangles*
▱ —*parallelogram*
▱s—*parallelograms*
⊙ —*circle*
⊙s—*circles*
∴ —*therefore.*

Ax.—*axiom.*
Cons.—*construction.*
Cor.—*corollary.*
Def.—*definition.*
Ex.—*exercise.*
Auth.—*authority.*
Hyp.—*hypothesis.*
Rt.—*right.*
Sch.—*scholium.*
St.—*straight.*
Sug.—*suggestion.*
Sugs.—*suggestions.*
P.—*Postulate.*

Q. E. D.—*Which was to be demonstrated.*

### Proposition I.*

**42. Theorem.** *Two straight angles are equal.*



**Let A O B and C S D represent two straight angles.**

*To prove that angle A O B is equal to angle C S D.*

*Suggestion 1.* What kind of a line do the two sides of $\angle A O B$ form? Of $\angle C S D$? § 18.

*2.* Place $\angle A O B$ so that point $O$ lies upon $S$, and another point of line $A O B$ upon line $C S D$. § 40, P. 2.

*3.* Where does $\angle A O B$ lie? *Ax.* 12.

*4.* How then does the amount of revolution in turning through $\angle A O B$ (§ 17 *c*), compare with that in turning through $\angle C S D$? *Ax.* 13.

*Therefore—*

What is the hypothesis in this theorem?

What is the conclusion?

What is the general enunciation? The special?

Ex. 1. How many straight lines can be drawn through three points not in a straight line?

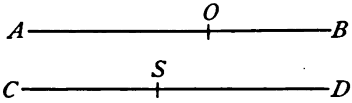
---

* Note.—The pupil is expected to study the suggestions carefully, to follow the directions when directions are given, to answer the questions when questions are asked, and to give the authority on which the answers are based; then to review the whole demonstration in a consecutive manner, without the aid of the suggestions. To illustrate what is expected of the pupil, model demonstrations are given of a few propositions, but no model should be consulted until after the proposition has been studied by means of the suggestions. The pupil will be more likely to avoid indefiniteness if he writes out all of his demonstrations for the first few weeks.

### Model.

### Proposition I.

**Theorem.** *Two straight angles are equal.*



**Let A O B and C S D represent two straight angles.**

*To prove that angle A O B is equal to angle C S D.*

The two sides of ∠ *A O B* and ∠ *C S D* form straight lines. § 18.

Place ∠ *A O B* upon ∠ *C S D* so that point *O* lies upon *S* and some other point of line *A O B*, as *A*, lies upon a part of line *C S D*. § 40, P. 2.

The lines *A O B* and *C S D* coincide, and therefore are one and the same line. *Ax.* 12.

∴ The ∠ *A O B* = ∠ *C S D*. *Ax.* 13, § 17 c.

*Therefore*—Two straight angles are equal. Q. E. D.

---

Ex. 2. How many straight lines can be drawn through four points no three of which are in the same straight line, if each line connects two of the four points?

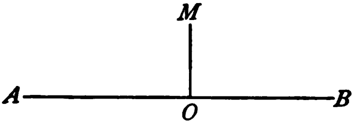
Ex. 3. What is the greatest number of points in which three straight lines can intersect?

Ex. 4. What is the greatest number of points in which four straight lines can intersect?

Ex. 5. If an angle is a right angle, what is its supplement?

## PROPOSITION II.

**43. Theorem.** *One straight angle is equal to two right angles.*



**Let A O B represent a straight angle.**

*To prove that the angular magnitude A O B is equal to two right angles.*

*Suggestion 1.* What kind of a line is $A\ O\ B$? § 18.

*2.* Let $M\ O$ represent a straight line that makes equal angles with the line $A\ B$. *Ax.* 14.

What kind of ∠s are $A\ O\ M$ and $M\ O\ B$? § 20.

*Therefore—*

**44.** A perpendicular is **erected to a line** when it is drawn perpendicular to the line from a point in the line.

**45.** A perpendicular is **dropped to a line** when it is drawn perpendicular to the line from a point without the line.

---

Ex. 6. If an angle is two-thirds of a right angle, what is its supplement?

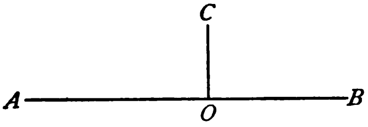
Ex. 7. If an angle is three-fourths of a right angle, what is its complement?

Ex. 8. The complement of angle $x$ equals one-third of its supplement. Find what part of a right angle $x$ is.

Ex. 9. The supplement of angle $x$ is two and one-half times its complement. Find $x$, the complement, and the supplement, each in terms of a right angle.

### PROPOSITION III.

46. **Theorem.** ***At any point in a straight line one perpendicular, and but one, can be erected.***



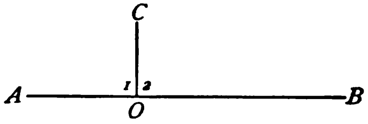
**Let A B represent a straight line and O any point in the line.**

FIRST.—*To prove that a perpendicular can be erected to A B at O.*

*Suggestion 1.* Draw *C O* to meet *A B* at *O*, making the two adjacent angles equal. *Ax.* 14.

2. See § 20 and 23.

SECOND.—*To prove that but one perpendicular can be erected to A B at O.*



*Suggestion 1.* Represent a perpendicular to *A B* at *O*, as *C O*. § 46, Part I.

2. How would $\angle$s *1* and *2* compare, if *C O* should be revolved either way, however little? Why? Ax. 15.

3. $\angle$s *1* and *2* would then be what kind of $\angle$s? Auth.

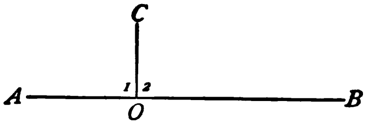
4. *C O* would be what kind of a line? § 24.

*Therefore—*

### Model.

### Proposition III.

**Theorem.** *At any point in a straight line one perpendicular and only one can be erected.*



**Let A B represent a straight line and O any point in that line.**

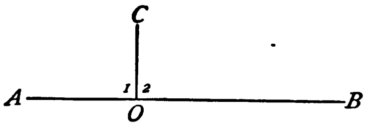
First.—*To prove that a perpendicular can be erected to A B at O.*

Let *C O* represent a line making two equal adjacent ∠s with *A B* at *O*, as ∠ 1 and ∠ 2. *Ax.* 14.

∠s 1 and 2 are right ∠s. § 20.

*C O* is ⊥ to *A B*. § 23.

*Therefore*—At a given point in a line a perpendicular can be erected to the line. Q. E. D.



Second.—*To prove that but one perpendicular can be erected to A B at O.*

**Let C O represent a perpendicular to A B at O.**

If $C O$ is revolved ever so little about the point $O$, $\angle 1$ would be greater or less than $\angle 2$. *Ax.* 15.

Hence $\angle$s 1 and 2 would be oblique $\angle$s. § 22.

$C O$ would be an oblique line. § 24.

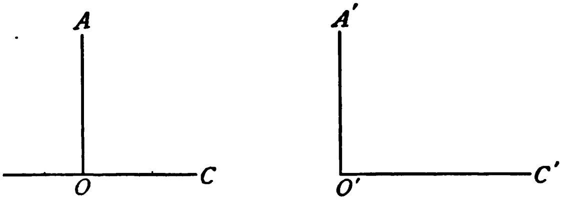
*Therefore*—Not more than one perpendicular can be erected to a line at a given point. Q. E. D.

47. Corollary.—Through the vertex of a given angle one and only one straight line can be drawn which bisects the given angle.

*Suggestion.*—See method used in demonstrating the theorem. *Axs.* 14 and 15.

## Proposition IV.

48. **Theorem.** *All right angles are equal.*



**Let A O C and A′ O′ C′ represent any two right angles.**

*To prove that angle $A O C$ and angle $A'O'C'$ are equal.*

*Suggestion 1.* Place $\angle A O C$ upon $\angle A'O'C'$ so that $A O$ lies upon $A'O'$, $O$ upon $O'$. § 40, P. 2.

*2.* Where does $O C$ fall? § 46.

*3.* How then do $\angle A O C$ and $\angle A'O'C'$ compare? *Ax.* 13.

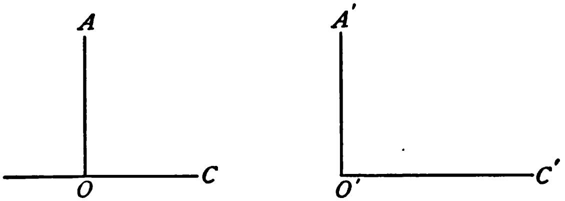
49. Any magnitude is **bisected** when it is divided into two equal parts.

50. Two figures **coincide** when each point of one lies in a corresponding point of the other.

### Model.

### PROPOSITION IV.

**Theorem.** *All right angles are equal.*



**Let A O C and A′ O′ C′ represent any two right angles.**

*To prove that angle A O C is equal to angle A′O′C′.*

Place ∠ *A O C* upon ∠ *A′O′C′*, *A O* upon *A′O′* with *O* upon *O′*. § 40, P. 2.

Line *O C* falls upon line *O′C′*. § 46.

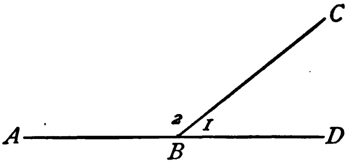
∠ *A O C* coincides with ∠ *A′O′C′*. § 50.

∴ Rt. ∠ *A O C* = Rt. ∠ *A′O′C′*. *Ax.* 13. § 17 (c).

*Therefore*—All right angles are equal. Q. E. D.

### PROPOSITION V.

**51. Theorem.** *If one straight line meets another straight line, the sum of the two adjacent angles formed is equal to two right angles.*



**Let C B and A D represent any two straight lines which meet at a point, as B, forming two adjacent angles, as angles 1 and 2.**

*To prove that the sum of angles 1 and 2 is equal to two right angles.*

First.—Suppose $\angle$s *1* and *2* are equal.

*Suggestion 1.* How many right angles are formed? § 20.

Second.—Suppose $\angle$s *1* and *2* are unequal.

*1.* What kind of an $\angle$ is $ABD$? § 18.

*2.* Compare $\angle 1 + \angle 2$ with $\angle ABD$. 17 c, *Ax.* 13.

*3.* See Proposition II. and complete the demonstration.

*Therefore—*

Query.—In Proposition V., what is the hypothesis? What is the conclusion?

**52.** Corollary 1.—The sum of all the angles on one side of a straight line, having a common vertex in the line, is equal to two right angles.

**53.** Corollary 2.—The total angular magnitude about a point is equal to four right angles.

---

Ex. 10. If the angular magnitude about a point is divided into six equal angles, each angle is what part of a right angle?

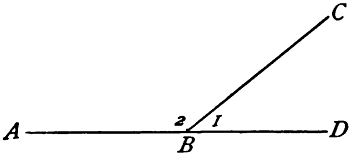
Ex. 11. If the angular magnitude about a point is divided into three angles, the second of which is twice the first, and the third is three times the first, how many right angles in each of the three angles?

Ex. 12. A line drawn perpendicular to the bisector of an angle at the vertex makes equal angles with the sides of the angle.

## Model.

### PROPOSITION V.

**Theorem.** *If one straight line meets another straight line, the sum of the two adjacent angles formed is equal to two right angles.*



**Let C B meet A D, at B.**

*To prove that the sum of angles 1 and 2 is equal to two right angles.*

FIRST.—If $\angle$s *1* and *2* are equal.

$\angle 1 + \angle 2 = 2$ Rt. $\angle$s. § 20.

SECOND.— If $\angle$s *1* and *2* are unequal.

$\angle 1 + \angle 2 =$ the st. $\angle ABD$. § 17 (c), *Ax.* 13.

The st. $\angle ABD = 2$ Rt. $\angle$s. § 43.

$\therefore \angle 1 + \angle 2 = 2$ Rt. $\angle$s. *Ax.* 1.

*Therefore*—If one straight line meets another straight line, the sum of the two adjacent angles is equal to two right angles.

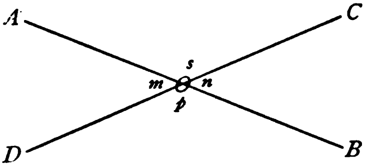
---

Ex. 13. In Fig. 12, if angle $COB$ is one-half of a right angle, angle $AOB$ equals what?

Ex. 14. In Fig. 14, if angle *1* is one-third of a right angle, and angle *4* is three-fourths of a right angle, angle $CBE$ equals what?

### PROPOSITION VI.

**54. Theorem.** *If two straight lines intersect, the vertical angles formed are equal.*



**Let A B and C D intersect at O, forming the vertical angles *m* and *n*, and *s* and *p*.**

*To prove that angle m is equal to angle n.*

*Suggestion 1.* $\angle m + \angle s =$ what? Why?

*2.* $\angle s + \angle n =$ what? Why?

*3.* Compare $\angle m + \angle s$ and $\angle n + \angle s$. Give auth.

*4.* Compare $\angle m$ and $\angle n$. *Ax.* 3.

*Therefore—*

In a similar manner compare $\angle s$ and $\angle p$.

---

Ex. 15. If there are three angles about a point, and one of them is equal to one and one-fifth right angles, and one to nine-tenths of a right angle, what is the magnitude of the other?

NOTE.—The student should make a careful study of the form and nature of a demonstration. In respect to form: first in order is the statement of the theorem or the **general enunciation**, this should be followed by the application of the theorem to a figure or the **special enunciation**, then follows the **proof**, and finally the **conclusion**.

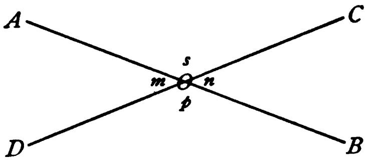
In the special enunciation each feature of the theorem should be carefully applied, close attention being paid, first, to what is given or known, and second, to what is to be determined. In the proof, each statement made should be based upon authority, which should consist of an axiom, a postulate, a definition, or a previously proved proposition. The student should stand ready to demonstrate all propositions used as authority. Care should be taken to see that these authorities exactly apply.

The order of procedure in an original demonstration depends upon the use that can be made of what is given, in arriving at what is to be determined.

## Model.

### PROPOSITION VI.

**Theorem.** *If two straight lines intersect, the vertical angles formed are equal.*



**Let AB and CD intersect at O, forming the vertical angles *m* and *n*.**

*To prove that angle m equals angle n.*

$\angle m + \angle s = 2$ Rt. $\angle$s. § 51.

$\angle n + \angle s = 2$ Rt. $\angle$s. § 51.

$\therefore \angle m + \angle s = \angle n + \angle s$. *Ax.* 1.

$\therefore \angle m = \angle n$. *Ax.* 3.

QUERY.—In this proposition, which is the special enunciation? Which the general? Which the hypothesis? Which the conclusion?

### Polygons.

**55.** A **polygon** is a portion of a plane bounded by straight lines; as *M N O*, etc., in Fig. 1.

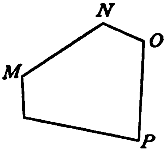


FIG. 1.

**56.** The bounding lines are the **sides** of the polygon, and their sum is the **perimeter** of the polygon.

**57.** The angles formed by the sides of the polygon on the side of the inclosed space, are the **interior angles** of the polygon, as angle *B A E* in Fig. 2.

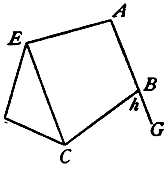


FIG. 2.

58. An angle formed by one side of the polygon and an adjacent side extended, is an **exterior angle** of the polygon; as angle *h* in Fig. 2.

59. The vertices of the interior angles of a polygon are the **vertices of the polygon.**

60. A straight line joining any two vertices, not adjacent, is a **diagonal** of a polygon; as *E C* in Fig. 2.

61. If there is no ambiguity, a polygon may be **read** by naming any two vertices not adjacent; as *A D*, *B D*, *M P*, etc.

62. Polygons are classified according to the **number of their sides.** The least number of sides a polgyon can have is three.

63. A polygon of three sides is a **triangle.**

64. A polygon of four sides is a **quadrilateral.**

65. A polygon of five sides is a **pentagon.**

66. A polygon of six sides is a **hexagon,** etc.

67. An **equilateral polygon** is a polygon all of whose sides are equal.

68. An **equiangular polygon** is a polygon all of whose angles are equal.

69. A **convex polygon** is a polygon no side of which, if extended, enters the space inclosed by the perimeter of the polygon; as *A B C*, etc., Fig. 2.

70. A **concave polygon** is a polygon, two or more sides of which, if extended, would enter the space inclosed by the perimeter of the polygon; as Fig. 3. If either *A B* or *B C* is extended through *B* it would enter the space inclosed by the perimeter of the polygon.

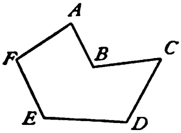


FIG. 3.

71. The angle $A B C$ in this figure is a **re-entrant angle.**

72. A **right angled triangle,** or a **right triangle,** is a triangle one of whose angles is a right angle.

(a) The side opposite the right angle is the **hypotenuse.**

(b) The other two sides are **the legs** of the triangle.

73. An **acute angled triangle,** or an **acute triangle,** is a triangle all of whose angles are acute.

74. An **obtuse angled triangle,** or an **obtuse triangle,** is a triangle one of whose angles is obtuse.

75. Acute or obtuse triangles are sometimes called **oblique triangles.**

76. A **scalene triangle** is a triangle no two sides of which are equal.

77. An **isosceles triangle** is a triangle which has two equal sides. The equal sides are the **legs** of the triangle.

78. A triangle which has all three sides equal is an **equilateral triangle.**

79. The **base** of a triangle is a selected side or the side upon which it is supposed to stand. The angle which is opposite the base is the **vertical angle** or the **vertex** of the triangle. Generally, any side may be taken as the base, but in an isosceles triangle that side which is not one of the two equal sides is always considered the base.

80. The **altitude** of a triangle is the perpendicular from the vertex to the base or the base extended.

NOTE.—The word altitude may refer to the line or to the length of the line, expressed in terms of some unit. The context will determine which use is intended.

81. The **median line,** or **median** of a triangle, is a line drawn from the vertex to the middle of the opposite side. $A B$ is the median, in Fig. 4.

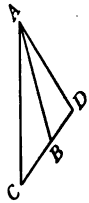
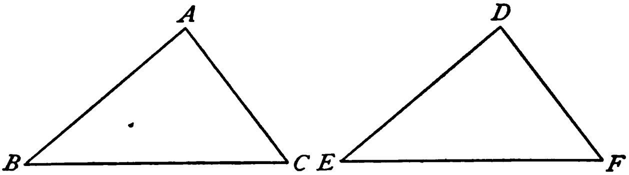


FIG. 4.

## PROPOSITION VII.

82. **Theorem.** *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*



**Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place $\triangle\, A\, B\, C$ upon $\triangle\, D\, E\, F$, so that point $B$ is upon $E$ and line $B\, C$ lies in $E\, F$. § 40 P.

Where does point $C$ fall? Why?

*2.* What direction does $B\, A$ take with respect to $E\, D$? § 17 (e).

3. Where does the point A lie? Why?

*4.* Points $C$ and $A$ being located, where, with respect to $D\, F$, does the line $A\, C$ lie? *Ax.* 12, Cor. 11.

*5.* What, now, is the position of $\triangle\, A\, B\, C$ with respect to $\triangle\, D\, E\, F$? § 50.

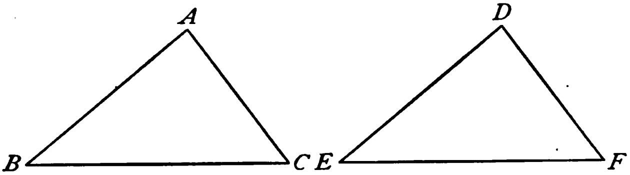
*6.* How, then, does $\triangle\, A\, B\, C$ compare with $\triangle\, D\, E\, F$? *Ax.* 13.

*Therefore—*

### Model.

### PROPOSITION VII.

**Theorem.** *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*



**Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

Place △ $A\ B\ C$ upon △ $D\ E\ F$, so that $B$ is upon $E$ and $B\ C$ lies in $E\ F$. § 40, P. 1.

$C$ falls upon $F$. $(B\ C = E\ F.)$

$B\ A$ takes the direction of $E\ D$. $(\angle B = \angle E.)$ § 17 (e).

$A$ falls upon $D$. $(B\ A = E\ D.)$

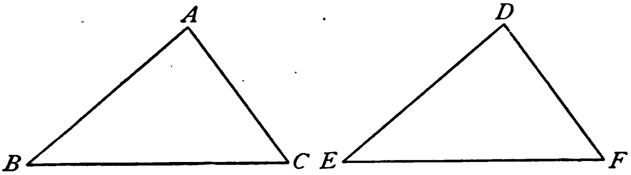
$A\ C$ coincides with $D\ F$. *Ax.* 12 Cor. 2.

$\therefore \triangle A\ B\ C = \triangle D\ E\ F$. *Ax.* 13. Q. E. D.

NOTE.—The method of proof in this proposition is the method of superposition, and consists in mentally placing one figure upon the other and finding that they exactly coincide.

### PROPOSITION VIII.

83. **Theorem.** *If two triangles have two angles and the included side of one, equal to two angles and the included side of the other, each to each, the triangles are equal in all respects.*



**Let ABC and DEF represent two triangles, having BC equal to EF, angle B equal to angle E, and angle C equal to angle F.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *A B C* upon △ *D E F*, so that *B* falls upon *E*, and *B C* on *E F*. Where does *C* fall? Why?

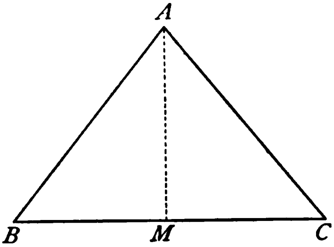
2. What direction does *B A* take? Why?
3. Where does *A* fall? Why? Sug. 2.
4. What direction does *C A* take? Why?
5. Where, now, does the point *A* fall? Sugs. 2 and 4.
6. Then how does △ *A B C* compare with △ *D E F?* *Ax.* 13.

*Therefore —*

84. SCHOLIUM. In equal triangles, equal angles lie opposite equal sides, and equal sides lie opposite equal angles.

## PROPOSITION IX.

**85. Theorem.** *The angles opposite the equal sides of an isosceles triangle are equal.*



**Let ABC represent an isosceles triangle, AB being equal to AC.**

*To prove that angle B is equal to angle C.*

*Suggestion 1.* Let $AM$ be drawn to represent a bisector of $\angle A$, and be extended until it meets $BC$, as at $M$.

2. Compare $\triangle ABM$ with $\triangle ACM$. § 82.
3. Compare $\angle B$ with $\angle C$. § 84.

*Therefore* —

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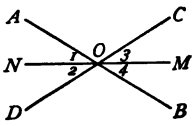
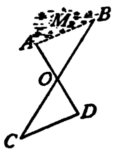


Ex. 16. If a straight line bisects one of a pair of vertical angles, prove that it bisects the other also.

If $MN$ bisects $\angle AOD$, prove that it bisects $\angle COB$; that is, that $\angle 3 = \angle 4$.

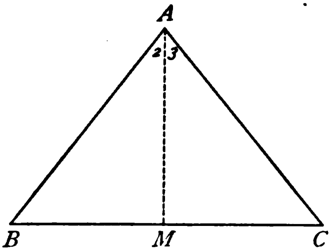


Ex. 17. Let $M$ represent a swamp or pond. Required to find the distance $AB$.

## Model.

### Proposition IX.

**Theorem.** *The angles opposite the equal sides of an isosceles triangle are equal.*



**Let ABC represent an isosceles triangle, AB being equal to AC.**

*To prove that angle B is equal to angle C.*

Let $AM$ be drawn to represent the bisector of $\angle A$, and be extended to meet $BC$, as at $M$.

In $\triangle$s $ABM$ and $ACM$,

$AB = AC$. Hyp.

$\angle 2 = \angle 3$. Cons.

$AM = AM$. *Ax.* 13.

$\therefore \triangle ABM = \triangle ACM$. § 82.

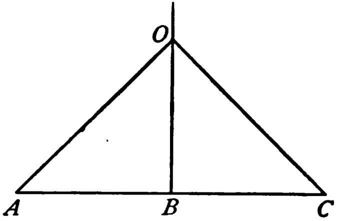
*Therefore:* $\angle B = \angle C$. § 84.

Q. E. D.

Ex. 18. If the equal sides of an isosceles triangle are extended beyond the base, prove that the exterior angles formed with the base are equal.

## PROPOSITION X.

**86. Theorem.** *If a perpendicular be erected at the middle point of a straight line, the distances from any point in the perpendicular to the extremities of the line are equal.*



**Let A C represent any line, B its middle point, B O a perpendicular to A C at B, and O any point in the perpendicular.**

*To prove that O A is equal to O C.*

*Suggestion 1.* In △s *O B A* and *O B C*, what parts are equal, each to each? Why?

*2.* How do △s *O B A* and *O B C* compare? § 82.

*3.* How, then, do *O A* and *O C* compare? § 84.

*Therefore* —

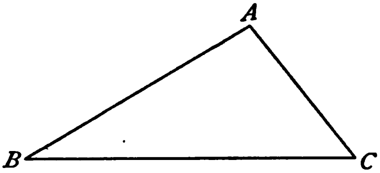
Ex. 19. Points, in the sides of an isosceles triangle, equidistant from the extremities of the base, are equidistant from the vertex.

Ex. 20. Prove that the line which bisects the vertical angle of an isosceles triangle bisects the triangle.

Ex. 21. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base. §s 23 and 20.

### PROPOSITION XI.

**87. Theorem.** *Any side of a triangle is less than the sum of the other two.*



**Let A B C represent any triangle.**

*To prove that any side, as A B, is less than the sum of the other two.*

*Suggestion:* Which represents the shorter distance from *A* to *B*, that by way of the line *A B*, or that by way of the lines *A C* and *C B?* Why? *Ax.* 11.

*Therefore* —

Ex. 22. If two straight lines intersect and one of the angles formed is a right angle, all of the angles are right angles.

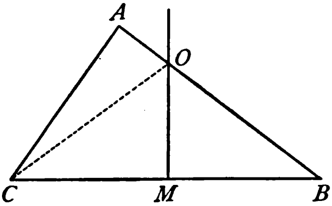
Ex. 23. A line which is perpendicular to the bisector of an angle makes equal angles with the sides of the angle. (1) if drawn through the vertex (Ex. 12); (2) if drawn through any other point of the bisector of the angle.

Ex. 24. Prove that an equilateral triangle is also equiangular.

Ex. 25. If the middle points of the sides of an isosceles triangle be joined by straight lines another isosceles triangle is formed. § 82.

## PROPOSITION XII.

**88. Theorem.** *If a perpendicular be erected at the middle point of a straight line, the distances from a point not in the perpendicular to the extremities of the line are unequal.*



**Let B C represent any straight line, M its middle point, O M a perpendicular to B C at the point M, A any point not in the perpendicular, and A B and A C lines drawn from A to the extremities of the line B C.**

*To prove that A B and A C are unequal.*

*Suggestion 1.* Let $O$ be the intersection of $A B$, and the perpendicular. Draw the line $O C$.

2. Compare $A C$ with $A O + O C$. § 87.
3. Compare $O C$ with $O B$. § 86.
4. Compare $O C + O A$ with $O B + O A$. *Ax.* 2.
5. Compare $A C$ with $A B$. *Ax.* 17.

*Therefore* —

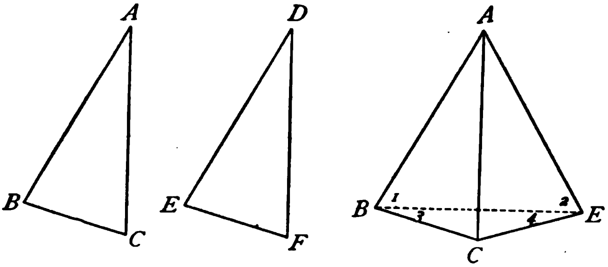
*Query:* Why draw the line $O C$?

Draw the figure so that $A C$ crosses $O M$, and prove the proposition.

Ex. 26. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other. *Ax.* 5.

## PROPOSITION XIII.

**89. Theorem.** *Two triangles, having the three sides of one equal, respectively, to the three sides of the other, are equal in all respects.*



**Let ABC and DEF represent two triangles, having AB equal to DE, AC equal to DF, and BC equal to EF.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *D E F* upon △ *A B C*, so that the longest side, *D F*, of △ *D E F*, coincides with the longest side, *A C*, of △ *A B C*, *D* upon *A*, and *F* upon *C*, but the point *E* upon the opposite side of *A C*, from *B*. Draw *B E*.

2. In △ *A B E*, compare *A B* with *A E*. Give auth
3. Compare ∠ 1 with ∠ 2. § 85.
4. In △ *C B E*, compare ∠3 with ∠ 4. Give auth.
5. Compare ∠ *B* with ∠ *E*. *Ax.* 2.
6. Compare △ *A B C* with △ *A E C*. § 82.
7. Then how does △ *A B C* compare with △ *D E F?*

*Therefore —*

**90.** The **premises** of a proposition are the conditions given upon which the conclusion is based.

91. The **converse** of a given proposition is a proposition which has the conclusion of the given proposition for one of the premises, and a premise of the given proposition for the conclusion.

Proposition *XIV.*, separated into premises and conclusion, may be stated:

*Premises:* The sum of two adjacent angles is equal to two right angles.

*Conclusion:* The exterior sides form a straight line.

Proposition *XIV.*, is the converse of Proposition *V.*, which, separated into premise and conclusion and interpreted in the language of Proposition *XIV.*, may be stated:

*Premises:* Two adjacent angles have a straight line for their exterior sides.

*Conclusion:* The sum of the angles is equal to two right angles.

92. When a proposition is proved to be true, it does not necessarily follow that its converse is also true.

---

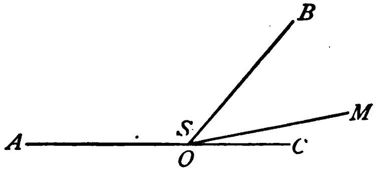
Ex. 27. If $A B C$ is an equilateral triangle, and $D$, $E$, and $F$ are points in the sides $A B$, $B C$, and $C A$, respectively, such that $A D$ is equal to $B E$ and to $C F$, prove that triangle $D E F$ is equilateral.

Ex. 28. If the straight line that joins the vertex of a triangle with the middle point of the base is perpendicular to the base, the triangle is isosceles.

Ex. 29. Prove that a line drawn from the vertex of an isosceles triangle to the middle of the base, (*1*) bisects the triangle, (*2*) bisects the vertical angle, (*3*) is perpendicular to the base.

## PROPOSITION XIV.

**93. Theorem.** *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*



**Let OA, OB, and OC be any three straight lines, which meet to form two adjacent angles S and BOC, whose sum is equal to two right angles.**

*To prove that the exterior sides, O A and O C, form a straight line.*

*Suggestion 1.* $AOC$ is either a straight line or a broken line. To determine which of these suppositions is true, draw an extension of $OA$, as $OM$.

*2.* How many rt. ∠s in $\angle S + \angle BOM$? Why?

*3.* How many rt. ∠s in $\angle S + \angle BOC$? Why?

*4.* Compare the sum of $\angle S + \angle BOM$, with the sum of $\angle S + \angle BOC$. Give auth.

*5.* Compare $\angle BOC$ with $\angle BOM$. Give auth.

*6.* Since the equal ∠s $BOC$ and $BOM$ have the common vertex $O$, and the common side $OB$, and since $OC$ and $OM$ are on the same side of $OB$, where does $OC$ lie with respect to $OM$? Why?

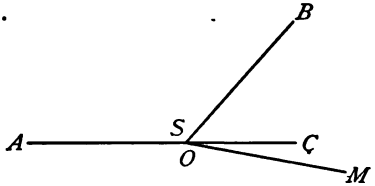
*7.* $OM$ in an extension of $OA$. Cons.
What relation does $OC$ sustain to $OA$?

*Therefore —*

### Model.

### Proposition XIV.

**Theorem.** *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*



**Let O A, O B, and O C, be any three straight lines, which meet to form two adjacent angles, S and B O C, whose sum is equal to two right angles.**

*To prove that the exterior sides, O A and O C, form a straight line.*

The line $A\ O\ C$ is either a straight line or a broken line.

Draw $O\ M$ an extension of $O\ A$.

$\angle S + \angle B O M$ = 2 rt. $\angle$s. § 51.

$\angle S + \angle B O C$ = 2 rt. $\angle$s. Hyp.

$\therefore \angle S + \angle B O M = \angle S + \angle B O C.$ *Ax.* 1.

$\therefore \angle B O M = \angle B O C.$ *Ax.* 3.

Since the equal $\angle$s $B O M$ and $B O C$ have a common vertex and the common side $O B$, and since $O C$ and $O M$ are on the same side of $O B$, $O C$ must lie upon $O M$. § 17 (e).

$A\ O\ M$ is a straight line. Cons.

$O\ C$ lies upon $O\ M$.

$\therefore A\ O\ C$ is a straight line.

*Therefore*—If the sum of two adjacent angles equals two right angles, their exterior sides form a straight line.

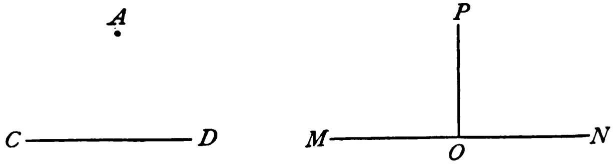
Ex. 30. Two angles are complements of each other, and the greater exceeds the less by 38 degrees. What are the angles?

Ex. 31. If two straight lines bisect each other at right angles, any point of either is equidistant from the extremities of the other. § 86.

Ex. 32. If the straight line that joins the vertex of a triangle with the middle point of the base is perpendicular to the base, the triangle is isosceles.

### PROPOSITION XV.

**94. Theorem.** *One perpendicular, and only one, can be dropped from a point to a line.*



**Let C D represent any straight line, and A any point without the line.**

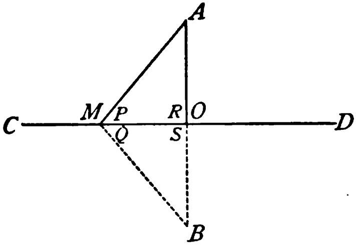
CASE I. *To prove that one perpendicular can be dropped from A to the line C D.*

*Suggestion 1.* Draw any straight line *M N*, and at any point of this line *O* erect the ⊥ *O P*. § 46.

2. Place the line *M N* upon the line *C D* and move it back and forth in *C D*. *O P* must at some time embrace point *A*.

Then *O P* is ⊥ to *C D* from point *A*.

*Therefore —*



**Let A O represent one perpendicular dropped from the point A to the line C D.** § 94 Case I.

Case II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

*Suggestion 1.* If another ⊥ can be dropped, let it be represented by $A\ M$.

*2.* Extend $O\ A$ to $B$, making $O\ B = O\ A$, and connect $M$ and $B$.

*3.* Compare $\triangle\ A\ O\ M$ with $\triangle\ B\ O\ M$. Give auth.

*4.* Compare $\angle\ P$ with $\angle\ Q$. Give auth.

*5.* If by construction $\angle\ P$ is a rt. $\angle$, what is $\angle\ Q$? Is line $A\ M\ B$ a straight or a broken line? Why? § 93.

*6.* Then how many straight lines are drawn from $A$ to $B$?

*7.* What, then, do you conclude about the statement that $A\ M\ B$ is a straight line? Why? *Ax.* 12, Cor. 2.

*8.* What do you conclude as to the possibility of $A\ M$ being a ⊥ from $A$ to $C\ D$? Why? Sug. 1.

*9.* Then, how many ⊥s can be dropped from a point to a straight line?

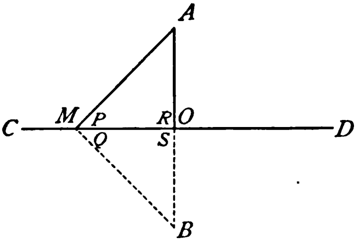
*Therefore* — ____________

Ex. 33. The supplement of an acute angle is how much more than the complement of the same acute angle?

### Model.

### PROPOSITION XV.

**Theorem.** *Only one perpendicular can be dropped from a given point to a line.*



**Let O A represent one perpendicular dropped from the point A to the line C D.**

CASE II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

If another $\perp$ can be dropped let it be represented by $A\,M$.

Extend $O\,A$ to $B$, so that $O\,A = O\,B$, and connect $M$ and $B$.

In $\triangle\ A\,O\,M$ and $B\,O\,M$, $O\,A = O\,B$. Cons.

$O\,M = O\,M$. Identical.

$\angle R = \angle S$. § 48.

$\therefore \triangle\ B\,O\,M = \triangle\ A\,O\,M$. § 82.

$\therefore \angle P = \angle Q$. § 84.

Since by hypothesis $P$ is a rt. $\angle$, $Q$ is a rt. $\angle$. *Ax.* 1.

$\therefore A\,M\,B$ is a straight line. § 93.

But $A\,O\,B$ is a straight line by construction.

$\therefore A\,M\,B$ is a broken line. *Ax.* 12, Cor. 2.

$\therefore A\,M$ is not $\perp$ to $C\,D$.

$\therefore$ But one $\perp$ can be dropped from a point to a line.

Q. E. D.

**95.** It is often convenient to express the magnitude of an angle in some other way than by using a right angle as the unit. To obtain another unit a right angle is divided into ninety equal parts, called **degrees.** The magnitude of an angle may, then, be expressed by stating how many degrees the given angle contains.

**96. Theorem.** *Complements of equal angles are equal.*

**97. Theorem.** *Supplements of equal angles are equal.*

NOTE.—In elementary Geometry only acute angles have complements, but either acute or obtuse angles may have supplements.

---

Ex. 34. If one of two supplementary adjacent angles is bisected, a perpendicular to the bisector through the vertex bisects the other angle.

Ex. 35. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplements of each other.

Ex. 36. How many degrees in a straight angle? In all the angular magnitude about a point?

Ex. 37. How many degrees in the supplement of two-thirds of a right angle?

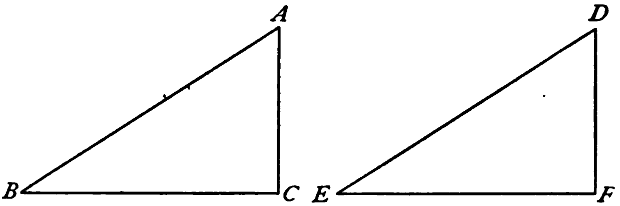
Ex. 38. How many degrees in an angle whose complement equals one-fourth of its supplement?

Ex. 39. The supplement of ten degrees is how much more than the complement of ten degrees?

Ex. 40. The straight lines bisecting the equal angles of an isosceles triangle and terminating in the sides, are equal. § 83.

### Proposition XVI.

98. **Theorem.** *Two right triangles which have the hypotenuse and an adjacent angle of one, equal to the hypotenuse and an adjacent angle of the other, each to each, are equal in all respects.*



**Let A B C and D E F represent two right triangles, having the hypotenuse A B equal to the hypotenuse D E, and angle A equal to angle D.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *A B C* upon △ *D E F*, so that *A B* coincides with *D E*, *A* upon *D*, and *B* upon *E*.

2. What direction does *A C* take? Why? § 17 (e).

3. Where does the point *C* fall? Why?

4. Since *B C* and *E F* are both ⊥ to the line *D F*, from point *E* where does *B C* lie? Why? § 94.

5. Where does point *C* fall? Why? Sugs. 2 and 4.

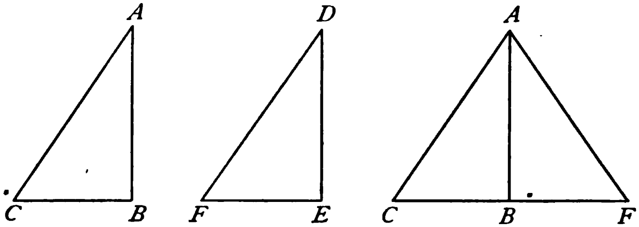
6. How, then, do the two △s compare? Why?

*Therefore* —

Note.—In the answer to suggestion 3, it will be seen that *C* must lie somewhere in the line *D F*, and, in the answer to Suggestion 5, *C* must lie somewhere in the line *E F*. Hence, in the answer to Suggestion 5, *C* can be exactly located.

## Proposition XVII.

**99. Theorem.** *Two right triangles which have the hypotenuse and a side of one, equal to the hypotenuse and a side of the other, each to each, are equal in all respects.*



**Let ABC and DEF represent two right triangles, having the hypotenuse AC equal to the hypotenuse DF, and the side AB equal to the side DE.**

*To prove that triangles* $ABC$ *and* $DEF$ *are equal in all respects.*

*Suggestion 1.* Place the triangles so that $AB$ coincides with $DE$, $A$ upon $D$, and $B$ upon $E$, with the vertices, $C$ and $F$, on opposite sides of $AB$.

*2.* Is $CBF$ a straight or broken line? Why? § 93.

*3.* What kind of an $\triangle$ is $ACF$? Why?

*4.* Compare $\angle$s $C$ and $F$. § 85.

*5.* Compare $\triangle$s $ABC$ and $ABF$, $ABC$ and $DEF$.

*Therefore* —

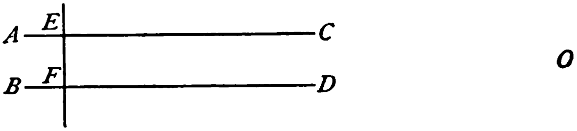
**100.** Name all the methods of determining the equality of triangles that have been demonstrated. Name those that relate to right triangles.

How many parts of a triangle have to be equal in order to make the triangles equal? Suppose the three angles of one triangle equal the three angles of another triangle, each to each, how do the triangles compare in equality?

**101.** Straight lines, in the same plane, that do not and cannot meet, however far extended, are **parallel lines.**

### PROPOSITION XVIII.

**102. Theorem.** *Two lines, which are perpendicular to the same line, are parallel.*



**Let A C and B D represent two lines, each perpendicular to the same line, E F, at the points E and F respectively.**

*To prove that A C and B D are parallel.*

*Suggestion 1.* $A$ $C$ and $B$ $D$ either meet or do not meet.

*2.* If they meet, let $O$ represent the point of meeting.

*3.* Compare the assumption that they meet at $O$ with § 94.

*4.* Do the lines $A$ $C$ and $B$ $D$ meet?

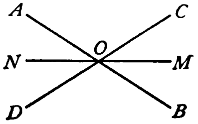
*5.* See § 101.

*Therefore* —

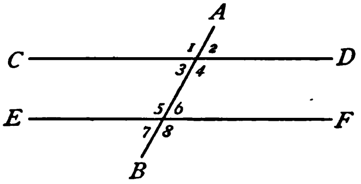
---

Ex. 41. Through two points an inch apart draw two parallel lines. *Sug.* Use corner of card or sheet of paper.

Ex. 42. If two vertical angles are bisected by two straight lines, prove that the bisectors together form one and the same straight line. Prove that $N$ $O$ $M$ is a straight line.



**103.** A **transversal** or **secant line** is a line which crosses two or more lines; as the line *A B*.



When a transversal cuts two lines eight angles are formed, *viz.*, the angles 1 to 8 in the figure.

**104.** The **interior angles** are those within, or between, the lines; as 3, 4, 5 and 6.

**105.** The **exterior angles** are those without the lines; as 1, 2, 7 and 8.

**106. Alternate interior angles** are pairs of non-adjacent interior angles on opposite sides of the transversal; as 3 and 6, 5 and 4.

**107. Alternate exterior angles** are pairs of non-adjacent exterior angles on opposite sides of the transversal; as 1 and 8, 2 and 7.

**108. Corresponding angles** are pairs of non-adjacent angles on the same side of the transversal, one exterior and one interior; as 2 and 6,4 and 8, etc.

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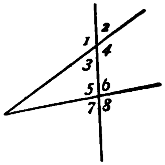
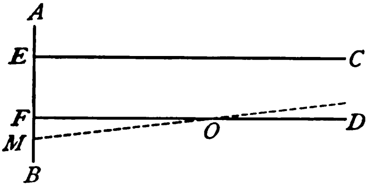


Ex. 43. Draw parallel and transversal lines so as to illustrate all kinds of angles that have been defined.

Ex. 44. If *D* is the middle point of the side *B C*, of triangle *A B C*, and *B E* and *C F* are the perpendiculars from *B* and *C* to *A D*, and *A D* extended, prove that *B E* is equal to *C F*. § 98.

### PROPOSITION XIX.

**109. Theorem.** *If one of two parallel lines is perpendicular to a given line, the other one is perpendicular to the same line.*



**Let E C and F D be two parallel lines, and let E C be perpendicular to A B.**

*To prove that F D is perpendicular to A B.*

*Suggestion 1.* From some point in *F D*, as *O*, draw *O M* ⊥ to *A B*.

*2.* What relation does *O M* sustain to *E C?* § 102.

*3.* What relation does *F D* sustain to *E C?* Hyp.

*4.* What relation does *M O* sustain to *F D?* *Ax.* 18.

*5.* Then what relation does *F D* sustain to *A B?* Why? Sug. 1.

*Therefore* —

Of what theorem is this the converse?

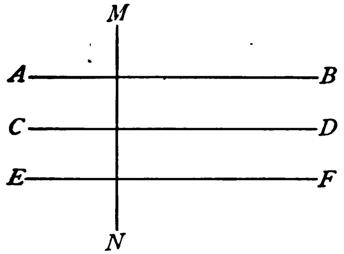
---

Ex. 45. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides, are equal.

Ex. 46. If two lines, *A B* and *C D*, intersect in the point *O*, and if the lines *A C* and *B D* be drawn, *A B* + *C D* is greater than *A C* + *B D*.

## Proposition XX.

**110 Theorem.** *If two lines are parallel to the same line, they are parallel to each other.*



**Let $AB$ and $CD$ each be parallel to $EF$.**

*To prove $AB$ and $CD$ are parallel.*

*Suggestion 1.* Draw $MN \perp$ to $EF$.

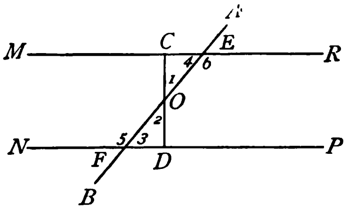
2. How is $MN$ related to $CD$? To $AB$? Why? § 109.

3. How are $AB$ and $CD$ related? Why? § 102.

*Therefore* —

## Proposition XXI.

**111. Theorem.** *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



**Let $MR$ and $NP$ represent two parallel lines cut by the transversal $AB$.**

Case I. *To prove that the alternate interior angles 4 and 3 are equal.*

*Suggestion 1.* Through $O$, the middle point of $EF$, draw $CD \perp$ to $NP$.

*2.* What relation does $CD$ sustain to $MR$? Why?

*3.* Compare $\triangle$s $OFD$ and $OEC$. Give auth.

*4.* Then, how do $\angle$s 4 and 3 compare? Why? § 84.

CASE II. *To prove that the alternate interior angles 6 and 5 are equal.*

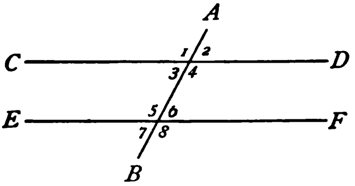
*Suggestion 1.* $\angle 4 + \angle 6 = \angle 5 + \angle 3$. Why?

*2.* Compare $\angle$s 6 and 5.

*Therefore —*

## PROPOSITION XXII.

**112. Theorem.** *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



**Let C D and E F represent two parallel lines, cut by the transversal AB.**

*To prove that the corresponding angles 2 and 6, or 3 and 7, etc., are equal.*

*Suggestion:* See § 111.

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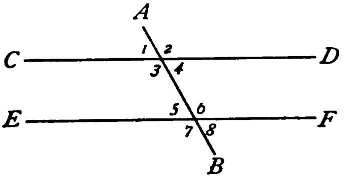
Ex. 47. In Proposition XIII., place two equal shorter sides upon each other, as $AB$ upon $DE$, connect $C$ and F, and demonstrate the proposition. *Ax. 3.*

### PROPOSITION XXIII.

113. **Theorem.** *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplements of each other.*



**Let C D and E F represent two parallel lines cut by the transversal A B, and let 4 and 6 be two interior angles on the same side of A B.**

*To prove that angles 4 and 6 are supplements of each other.*

*Suggestion 1.* Compare ∠s 3 and 6. Give auth.

*2.* Compare ∠s 3 and 4. § 28.

*3.* How, then, does ∠ 6 compare with ∠ 4? Why?

*Therefore* —

NOTE.—Several different methods of demonstration should be worked out in Propositions *XXII* and *XXIII*.

---

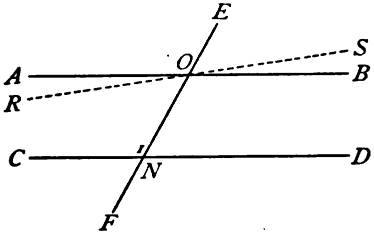
Ex. 48. The line joining the vertices of two isosceles triangles, on opposite sides of the same base, bisects the base and is perpendicular to it.

*Suggestion:* Method used in § 89.

Ex. 49. If a perpendicular is dropped from the vertex of an isosceles triangle to the base, prove (1) that it bisects the base; (2) that it bisects the vertical angle; and (3) that it bisects the triangle.

## Proposition XXIV.

**114. Theorem.** *If two straight lines are cut by the transversal, so that the alternate interior angles are equal, the lines are parallel.*



**Let A B and C D be two straight lines, cut by the transversal E F, so that angles B O F and 1 are equal.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through *O*, draw *R S* ∥ to *C D*.

*2.* Compare ∠ *S O F* with ∠ 1. Give auth.

*3.* Compare ∠ *B O F* with ∠ 1. Give auth.

*4.* Compare ∠ *S O F* with ∠ *B O F*. Give auth.

*5.* Since ∠s *S O F* and *B O F* are superimposed and have one side and the vertex common, what relation must *O S* and *O B* sustain to each other? § 17 (e).

*6.* Since *R S* is, by construction, ∥ to *C D*, what relation does *A B* sustain to *C D?* Why?

*Therefore* —

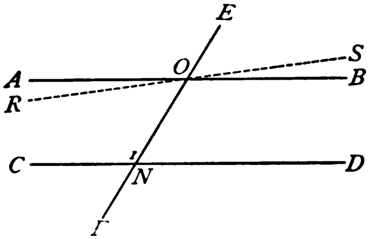
Of what proposition is this the converse?

---

Ex. 50. If two parallel lines are cut by a transversal, prove that the alternate exterior angles are equal.

### Model.

### PROPOSITION XXIV.



**Theorem.** *If two straight lines are cut by a transversal, so that the alternate interior angles are equal, the lines are parallel.*

**Let AB and CD be two straight lines, cut by the transversal EF, so that angles BOF and 1 are equal.**

*To prove that A B and C D are parallel.*

Through $O$, draw $RS$ to represent a line ‖ to $CD$.

The $\angle SOF = \angle 1$. § 111.

$\angle BOF = \angle 1$. Hyp.

$\therefore \angle SOF = \angle BOF$. *Ax.* 1.

$\angle SOF = BOF$ and $OF$ is common, therefore $OS$ and $OB$ coincide. § 17 (e).

But $OS$ is ‖ to $CD$, hence $AB$ is ‖ to $CD$. Q. E. D.

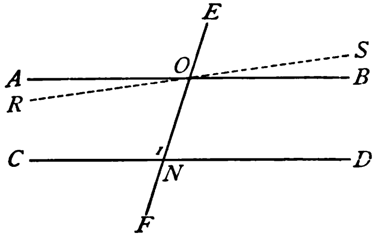
### Review.

If parallel lines are cut by a transversal, what pairs of angles are equal?

How may we know lines are parallel? Name two ways.

## PROPOSITION XXV.

115. **Theorem.** *If two straight lines are cut by a transversal so that the corresponding angles are equal, the two straight lines are parallel.*



**Let A B and C D represent two straight lines cut by the transversal E F so that the corresponding angles E O A and 1 are equal.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through *O*, draw *R S* to represent a line ‖ to *C D*.

*2.* Compare ∠ *E O R* with ∠ 1. Give auth.

*3.* Compare ∠ *E O A* with ∠ 1. Give auth.

*4.* Compare ∠ *E O R* with ∠ *E O A*. Give auth.

*5.* What relation does *O A* sustain to *O R*?

*6.* Since *R S* is, by construction, ‖ to *C D*, what relation does *A B* sustain to *C D*? Why?

*Therefore* —

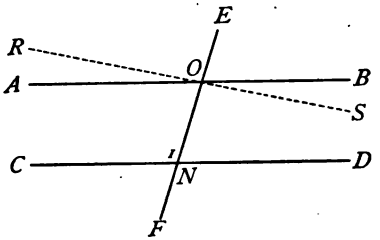
*Query:* If ∠ *1* is a rt. ∠, what previous theorem does *XXV* become?

---

Ex. 51. In the figure for § 112, compare angles 1 and 6; also angles 4 and 7.

## PROPOSITION XXVI.

**116. Theorem.** *If two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are the supplements of each other, the two straight lines are parallel.*



**Let A B and C D represent two straight lines cut by the transversal E F so that angles A O F and E N C are supplements of each other.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through *O*, draw *R S* to represent a line ‖ to *C D*.

2. Compare ∠ *A O F* with ∠ *R O F*. (See method in § 115.)

Complete the demonstration.

*Therefore* —

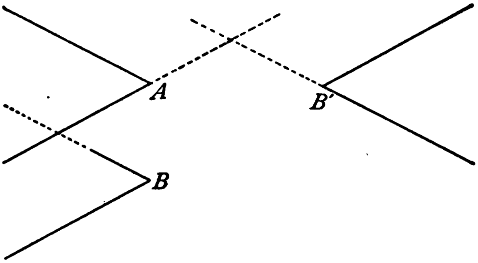
*Query:* If you were to construct a line parallel to a given line, how would you do it?

---

Ex. 52. In Fig. for § 103, if angle 1 contains 47½ degrees, how many degrees in angle 4? in angle 2? in angle 8? in angle 6? in angle 5? if angle 3 contains 39 degrees, how many degrees in angle 6? in angle 8? in angle 7? in angle 5? Give authority for each statement.

### PROPOSITION XXVII.

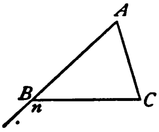
**117. Theorem.** *Two angles which have their sides respectively parallel and extending in the same direction, or in opposite directions, from their vertices are equal.*



**Let A and B represent two angles whose sides are respectively parallel and extend in the same direction from their vertices; and A and B′ two angles whose sides are respectively parallel and extend in opposite directions from their vertices.**

*To prove angles A and B equal, also angles A and B′.*

*Suggestion:* Extend a side of one angle until it meets a side, or an extended side of the other. Complete the demonstration.



**118. Opposite interior angles** of a triangle are the two angles of the triangle not adjacent to the exterior angle of the triangle, as ∠s $A$ and $C$. ∠ $n$ is an exterior ∠. § 58.

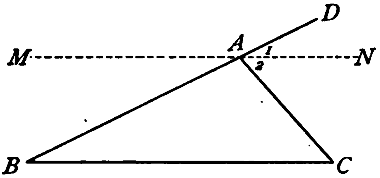
---

Ex. 53. The perimeter of a triangle is less than twice the sum of the medians.

Ex. 54. In what kind of a triangle does the bisector of the vertical angle coincide with both the median to the base and the perpendicular from the vertex to the base?

### PROPOSITION XXVIII.

**119. Theorem.** *An exterior angle of a triangle is equal to the sum of the opposite interior angles.*



**Let ABC represent a triangle, DAC an exterior angle, and B and C the opposite interior angles.**

*To prove that angle D A C is equal to the sum of angles B and C.*

*Suggestion 1.* Through the vertex *A* draw a line *M N* ‖ to *B C*.

*2.* Compare ∠ 1 with ∠ *B*. Give auth.

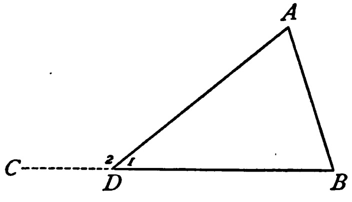
*3.* Compare ∠ 2 with ∠ *C*. Give auth.

*4.* Compare ∠ *D A C* with ∠ *B* + ∠ *C*. Give auth.

*Therefore* —

### PROPOSITION XXIX.

**120. Theorem.** *The sum of the interior angles of a triangle is equal to two right angles*



**Let ABD represent a triangle.**

*To prove that the sum of angles A, B, and 1 is equal to two right angles.*

*Suggestion 1.* Extend one of the sides, as *B D*.

*2.* Compare ∠s 1 + 2 with ∠s 1 + *A* + *B*.

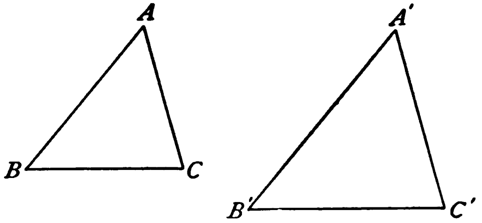
*Therefore* —

**121.** COROLLARY I. *A triangle can have only one obtuse angle.*

**122.** COROLLARY II. *Every right triangle has two acute angles, each of which is the complement of the other.*

## PROPOSITION XXX.

**123. Theorem.** *If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.*



**Let ABC and A′B′C′ represent two triangles, having angles A and A′ equal, also angles B and B′.**

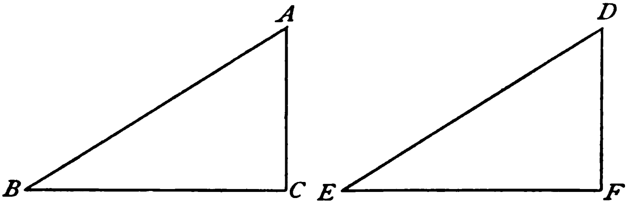
*To prove that angles C and C′ are equal.*

*Suggestion:* See § 120.

**124.** COROLLARY. *If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.*

### PROPOSITION XXXI.

**125. Theorem.** *If two right triangles have a side and an acute angle of one equal to the corresponding side and acute angle of the other respectively, the right triangles are equal in all respects.*



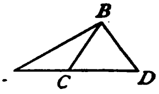
**Let ABC and DEF represent two right triangles, in which AC is equal to DF, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion:* Compare ∠s *A* and *D*. § 123.
Complete the demonstration.
*Therefore* —

---

Ex. 55. Two triangles having two sides and an angle opposite one of them equal each to each may or may not be equal.

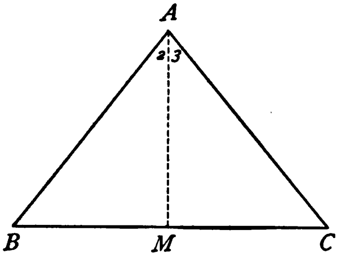


*Suggestion:* If the equal ∠s are acute and opposite the shorter of the two equal sides, the Δs may not be equal.

Study the figure and show this to be true. Show the equality of the Δs in other cases. $BD = BC$.

## Proposition XXXII.

**126. Theorem.** *If two angles of a triangle are equal, the sides opposite them are equal and the triangle is isosceles.*



**Let $ABC$ represent a triangle in which the angle $B$ is equal to angle $C$.**

*To prove that the side $AC$ is equal to the side $AB$.*

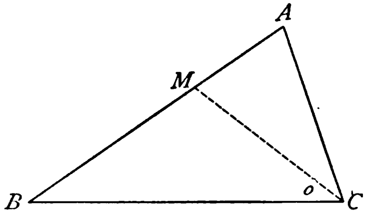
*Suggestion 1.* Drop a $\perp$ from $A$ to $BC$, as $AM$.
*2.* Compare $\triangle$s $AMB$ and $AMC$. Give auth.
*3.* Compare $AC$ with $AB$.
*Therefore* —

---

Ex. 56. If $ABC$ is a right triangle with the right angle at $C$, and if through $C$ a line meeting the hypotenuse at $D$ is drawn in such a manner that angle $ACD$ equals angle $A$, prove (1)that $CD$ bisects the hypotenuse $AB$, and (2) that if angle $ACD$ is not equal to $A$, $CD$ does not bisect the hypotenuse.

### PROPOSITION XXXIII.

**127. Theorem.** *If two angles of a triangle are unequal, the sides opposite them are unequal, and the greater side is opposite the greater angle.*



**Let ABC represent a triangle in which the angle C is greater than the angle E.**

*To prove that the side B A is greater than the side C A.*

*Suggestion 1.* Draw $C M$ to represent a line making $\angle o$ equal to $\angle B$. Is this possible?

2. Compare $B M$ and $C M$. Give auth.

3. Compare $C M + M A$ with $C A$, $B A$ with $C A$. Give auth.

*Therefore* —

---

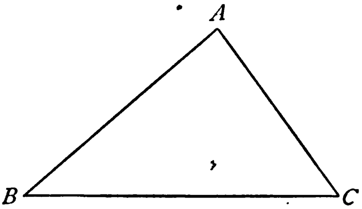
Ex. 57. Prove Proposition XXIX by other methods.

*Suggestion:* Through $A$ draw a line ∥ to $D B$. Or extend $D A$ and $B A$ through $A$, and draw through $A$ a line ∥ to $D B$.

Ex. 58. If $A B C$ is a right triangle with the right angle at $C$, and if through $C$ a line meeting the hypotenuse at $D$ is drawn in such a manner that angle $A C D$ equals angle $B$, prove that $C D$ is perpendicular to the hypotenuse $A B$.

### PROPOSITION XXXIV.

**128. Theorem.** *If two sides of a triangle are unequal the angles opposite them are unequal, and the greater angle is opposite the greater side.*



**Let A B C represent a triangle in which the side A B is greater than the side A C.**

*To prove that angle C is greater than angle B.*

*Suggestion 1.* $\angle C$ equals $\angle B$, is less than $\angle B$, or is greater than $\angle B$.

*2.* If $\angle C = \angle B$, compare $AB$ and $AC$. Can $\angle C = \angle B$? § 126.

*3.* If $\angle$ C is greater than $\angle B$, compare $AB$ and $AC$. § 127.

Can $\angle$ C be greater than $\angle$ B?

*4.* If $\angle C$ is less than $\angle B$, compare $AB$ and $AC$.

Can $\angle C$ be less than $\angle B$?

*Therefore* —

Of what proposition is this the converse?

Compare propositions *IX*, *XXXII*, *XXXIII*, and *XXXIV*.

---

Ex. 59. If from any point in the base of an isosceles triangle perpendiculars are dropped to the sides, these perpendiculars make equal angles with the base.

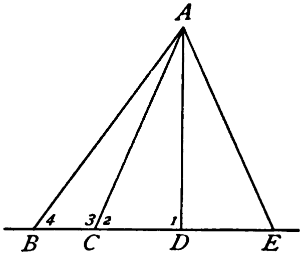
## PROPOSITION XXXV.

129. **Theorem.** *If a perpendicular and oblique lines are drawn from a point to a given line:*

CASE I. *The perpendicular is shorter than any oblique line.*

CASE II. *Two oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.*

CASE III. *Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the more remote is the greater.*



**Let A D be perpendicular to a given line B E, and A B, A C and A E oblique lines, meeting the given line at B, C and E respectively; and let D C be equal to D E and D B greater than D E.**

CASE I. *To prove that A D is shorter than any oblique line from A to B E.*

*Suggestion 1.* How does $\angle 1$ compare with $\angle 2$? Why?

2. How does $A D$ compare with $A C$? Why?

*Therefore* —

Case II. *To prove that A C equals A E.*

*Suggestion 1.* Compare triangles *A D C* and *A D E.*

*2.* Compare *A C* and *A E.* Give auth.

*Therefore —*

Case III. *To prove that A B is greater than A C or A E.*

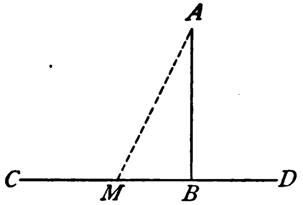
*Suggestion 1.* Compare ∠s 3 and 2; ∠s 3 and 4.

*2.* Compare lines *A B* and *A C.* Give auth.

*Therefore —*

### Proposition XXXVI.

**130. Theorem.** *A perpendicular is the shortest line that can be drawn from a point to a straight line.*



**Let A B represent the shortest line from A to the line C D.**

*To prove that A B is perpendicular to C D.*

*Suggestion 1.* If *A B* is not ⊥ to *C D*, draw *A M* to represent a ⊥ from *A* to *C D.*

*2.* ∴ *A M* is the shortest line from *A* to *C D.* Why? § 129, Case I.

*3.* What relation does *A M* sustain to *A B? Ax.* 16, Hyp.

*4.* What relation does *A B* sustain to *C D ?*

*Therefore —*

Of what theorem is this the converse? Compare them.

**131.** The **distance from a point to a line** is the length of the perpendicular from the point to the line.

**132.** Propositions *XXXIV* and *XXXVI* are good illustrations of what is known as the indirect method, or the ***reductio ad absurdum*** method of reasoning. Its peculiarity consists in the fact that the statement of the proposition is not directly proved to be true, but that everything which contradicts the statement of the proposition is shown to lead to some manifest absurdity, and is therefore false. This method often presents difficulties to the beginner on account of the fact that he is obliged to admit, temporarily, and for argument's sake, something which the argument itself goes to destroy. The indirect method is employed in demonstration of the converse of propositions. In applying this method care must be taken that every possible case which contradicts the proposition is considered, and each one shown to lead to an absurdity. Then, and then only, is this method of demonstration rigid.

NOTE.—If the student, in original investigation of the above propositions, or any similar one, should chance to consider the supposition which leads to the truth before one or more of the others, the remaining suppositions should be investigated. When all possible suppositions have been stated, one of which is true, and it has been shown that all but one are false, it is evident that the one remaining must be true.

Ex. 60. The line joining the feet of the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides, is parallel to the base.

Ex. 61. The base of an isosceles triangle, together with the bisectors of the angles at the base, form a second isosceles triangle.

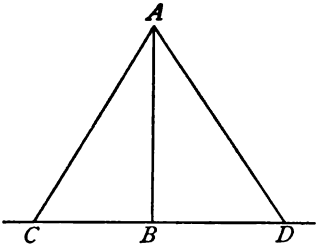
Ex. 62. Two angles having their sides respectively parallel, and each angle, having one side extending in the same direction from their vertices and the other side in opposite directions, are supplements of each other.

Ex. 63. If two triangles have two angles and a side opposite one of them equal respectively to two angles and a corresponding side of the other, the triangles are equal in all respects.

Ex. 64. Two isosceles triangles having equal bases and equal vertical angles are equal.

## PROPOSITION XXXVII.

133. **Theorem.** *Two equal oblique lines, drawn from the same point in a perpendicular to a given line, cut off equal distances from the foot of the perpendicular.*



**Let A B represent a perpendicular to C D, and let A C and A D represent equal oblique lines drawn from A to C D.**

*To prove that B C is equal to B D.*

Of what theorem is this the converse?

*Therefore—*

NOTE.—The pupil should try frequently to demonstrate a proposition with no drawing to assist him. If in recitation he can construct the figure in his mind and make clear his thought to the class, he is worthy of extra credit. Try it with proposition *XXXVII.*

*Queries —*

1. By how many methods can we determine that two lines are parallel? §§ 101, 102, 114, etc.

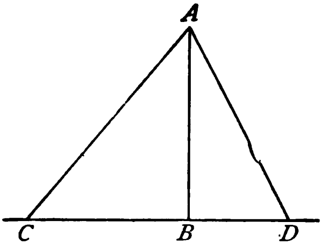
2. By how many methods can we know a line is perpendicular to a line? §§ 23, 109, Ex. 48.

---

Ex. 65. *M* and *N* are two parallel lines. Line *A* is a perpendicular to *M* and line *B* is perpendicular to *N*. Prove that *A* and *B* are parallel.

## PROPOSITION XXXVIII.

134. **Theorem.** *Two unequal oblique lines, drawn from the same point in the perpendicular to a given line, cut off unequal distances from the foot of the perpendicular, the longer line cutting off the greater distance.*



**Let AB represent a perpendicular to CD, and AC and AD unequal oblique lines drawn from A to CD, AC being longer than AD.**

*To prove that $BC$ is greater than $BD$.*

*Suggestion 1.* $BC = BD$, is less than $BD$, or is greater than $BD$.

*2.* If $BC = BD$, how would $AC$ compare with $AD$? Why?

*3.* If $BC$ is less than $BD$, how would $AC$ compare with $AD$? Give auth.

*4.* How, then, must $BC$ compare with $BD$?

*Therefore* — § 132.

Of what theorem is this the converse?

---

Ex. 66. Each angle of an equilateral triangle is one-third of two right angles, or two-thirds of one right angle.

### Locus of a Point.

**135.** To definitely locate a point in a plane, two conditions limiting its position must be known.

If only one condition is given, the point is to some extent, but not completely, determined. For example, if a point is at a given distance from some fixed point, it is not exactly located, but may move, provided that in its movement it always satisfies the requirement of being at a given distance from the fixed point. The moving point may occupy any position whatever in a line, which will later be defined as the circumference of a circle with the fixed point as the center.

**136.** When the position of a point in a plane is limited to and may be anywhere in a line or group of lines, the line or group of lines is the **locus of the point.**

The locus of a point is both inclusive and exclusive. It must include all the points that satisfy the given condition and exclude all that do not satisfy that condition. For example, if one would know that a circumference or any other line or group of lines is a locus, one must ascertain two things: first, that the point **may be anywhere** in the line or set of lines; second, that the point **must be somewhere** in the line or set of lines; i. e., it cannot be anywhere outside of them.

---

Ex. 67. A straight line drawn from any point in the bisector of an angle to either side and parallel to the other side, makes, with the bisector and the side to which the line is drawn, an isosceles triangle.

## PROPOSITION XXXIX.

**137. Problem.** *To determine the locus of a point at equal distances from the extremities of a given line.*

**Let A B represent a given line.**

*To determine the locus of a point at equal distances from A and B.*

*Suggestion 1.* The problem means that we are to find one or more lines, such that, *first*, any point in them is equally distant from *A* and *B*, and *second*, that no point without them is equally distant from *A* and *B*.

*2.* What line is everywhere equally distant from the extremities of a given line? § 86.

*3.* Compare the distances of any points without the $\perp$ from the extremities. § 88.

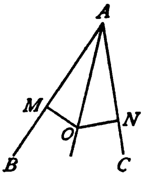
*4.* What is the required locus?

---

Ex. 68. Prove that every point in the bisector of an angle is equally distant from the sides of the angle.

*Suggestion: O M* and *O N* are $\perp$ to *A B* and *A C* respectively. § 131.

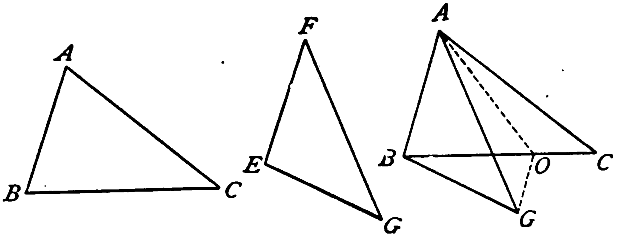
Prove *O M* equal to *O N*.



Ex. 69. If a line intersects the sides of an isosceles triangle at equal distances from the vertex, the line is parallel to the base.

### PROPOSITION XL.

138. **Theorem.** *If two triangles have two sides of one equal to two sides of the other and the included angles unequal, the remaining sides are unequal, and that side is the greater which is opposite the greater included angle.*



**Let ABC and FEG represent two triangles in which AB is equal to FE, AC is equal to FG, and angle A is greater than angle F.**

*To prove that BC is greater than EG.*

*Suggestion 1.* Place △ $FEG$ upon △ $ABC$, so that $FE$ coincides with $AB$, $F$ upon $A$, and $E$ upon $B$.

*2.* Since ∠ $F$ is less than ∠ $A$, where does $FG$ lie with respect to ∠ $A$? § 17 (e).

*3.* Bisect ∠ $GAC$ and extend to meet $BC$, as at $O$. Connect $O$ and $G$.

*4.* Compare △s $AOG$ and $AOC$.

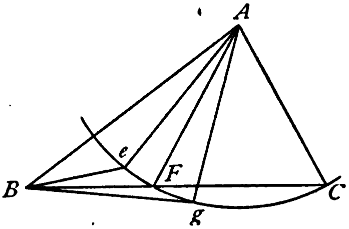
*5.* Compare lines $OG$ and $OC$.

*6.* Compare $BG$ with $BO + OG$; $BG$ with $BC$.

*Therefore* —

In placing $\triangle FEG$ upon $\triangle ABC$, $G$ may be without the $\triangle ABC$, as at $g$, upon line $BC$, as at $F$, or within $\triangle ABC$, as at $e$, as illustrated by the accompanying diagram.

Apply the demonstration above to the figure, with $G$ in each of the three positions.



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70. Prove that every point not in the bisector of an angle is unequally distant from the sides of the angle.

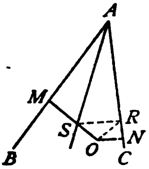
Let $O$ be at any point not in the sides of the angle.

Prove that $OM$ and $ON$ are unequal.

*Suggestion 1.* Draw $SR \perp$ to $AN$.

*2.* Connect $R$ and $O$.

*3.* Compare $SO + SR$ with $OR$, $OM$ with $OR$, $OM$ with $ON$.



Ex. 71. What is the locus of a point equally distant from the sides of an angle? Exs. 68 and 70.

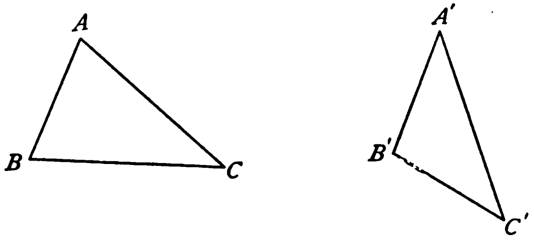
Ex. 72. What is the locus of a point at equal distances from the two intersecting lines? Ex. 71.

Ex. 73. The middle point of the hypotenuse of a right triangle is at equal distances from the three vertices.

*Suggestion:* Ex. 56, or drop $\perp$s from the middle point of the hypotenuse to the legs of the $\triangle$ and compare the triangles formed.

### PROPOSITION XLI.

**139. Theorem.** *If two triangles have two sides of one equal to two sides of the other and the third sides unequal, the angles opposite the third sides are unequal, and that side is the greater which is opposite the greater side.*



**Let A B C and A′ B′ C′ represent two triangles having A B equal to A′ B′, A C equal to A′ C′, and B C greater than B′ C′.**

*To prove that angle A is greater than angle A′.*

*Suggestion 1.* If $\angle A = \angle A'$, how do $BC$ and $B'C'$ compare? Why?

*2.* If $\angle A < \angle A'$, how do $BC$ and $B'C'$ compare? Why?

*3.* How, then, must $\angle A$ compare with $\angle A'$?

*Therefore —*

### Quadrilaterals.

**140. Quadrilaterals** are divided into classes, as follows:

(a) The **trapezium,** which has no two sides parallel. Fig. 1.

(b) The **trapezoid,** which has two sides parallel. Fig. 2.

(c) The **parallelogram,** which has its pairs of opposite sides parallel. Fig. 3.

**141.** The parallel sides of a trapezoid are the **bases,** the non-parallel sides are the **legs,** and the line perpendicular to the bases is the **altitude.**

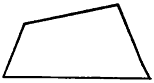


FIG. 1.

In Fig. 2, $A\ B$ and $C\ D$ are the bases, $A\ C$ and $B\ D$ are the legs, and $A\ M$ is the altitude of the trapezoid.

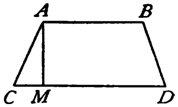


FIG. 2.

**142.** Any side of a parallelogram may be selected as the base. It is then called the **primary base.** The side opposite the primary base is the **secondary base.** The other sides of the parallelogram are its **sides** or **legs.**

**143.** The **altitude** of a parallelogram is a line perpendicular to the bases, as, $A\ B$ in Fig. 3.

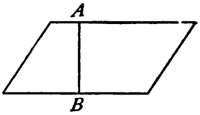


FIG. 3.

NOTE.—Usually the words **lower base** and **upper base** are used instead of primary base and secondary base, respectively, but as Geometry does not take into account the idea, up and down, the terms primary and secondary are preferable.

**144.** An oblique-angled parallelogram is a **rhomboid.** Fig. 3.

**145.** A right-angled parallelogram is a **rectangle.** Fig. 4.

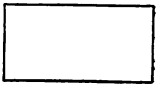


FIG. 4.

**146.** An equilateral rhomboid is a **rhombus.** Fig. 5.

**147.** An equilateral rectangle is a **square.** Fig. 6.

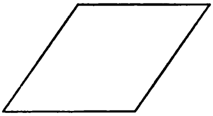


FIG. 5.

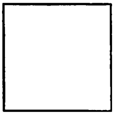
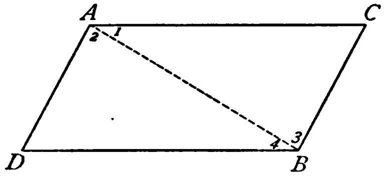


FIG. 6.

The student should make out his own classification of quadrilaterals.

### PROPOSITION XLII.

**148. Theorem.** *The opposite sides and opposite angles of a parallelogram are respectively equal, and adjacent angles are supplements of each other.*



**Let A C B D represent a parallelogram.**

*To prove that A C equals D B; C B equals A D; that angle A and angle B, angle C and angle D are respectively equal; and that the adjacent angles A and C, C and B, B and D, and D and A are respectively supplementary.*

*Suggestion 1.* Draw the diagonal *A B*, and compare $\Delta$s *A B D* and *A B C*.

2. How do *A C* and *B D* compare? *A D* and *C B*? Complete the demonstration.

*Therefore —*

**149.** COROLLARY 1. *The diagonal of a parallelogram divides it into two equal triangles.*

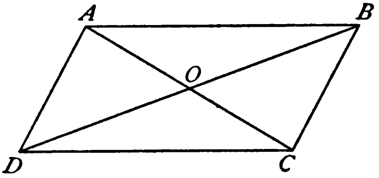
---

Ex. 74. The sum of the exterior angles at the base of any triangle is equal to two right angles plus the vertical angle of the triangle.

Ex. 75. In a triangle *A B C*, the angle *C* is twice the sum of angles *A* and *B*, and angle *B* is twice angle *A*; find all three angles of the triangle.

## Proposition XLIII.

**150. Theorem.** *The diagonals of a parallelogram bisect each other.*



**Let AC represent a parallelogram, and AC and BD the diagonals.**

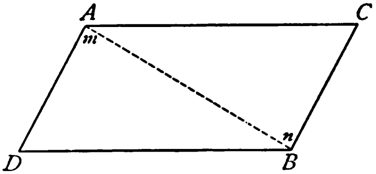
*To prove that A C and D B bisect each other.*

*Suggestion.* Prove by comparison of triangles.

*Therefore —*

## Proposition XLIV.

**151. Theorem.** *If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.*



**Let ACBD represent a quadrilateral in which the side A C is equal and parallel to the side B D.**

*To prove that A C B D is a parallelogram.*

*Suggestion 1.* What is the definition of a ▱?

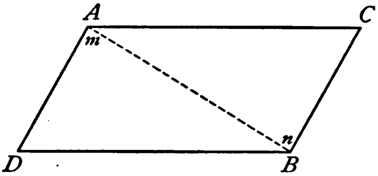
*2.* How much of the definition is given in the Hyp.? What remains to be proved?

*3.* Draw the diagonal $A\,B$. Compare △s $A\,B\,C$ and $A\,B\,D$. Compare $\angle m$ and $\angle n$. Complete the demonstration.

*Therefore* —

## PROPOSITION XLV.

**152. Theorem.** *A quadrilateral whose opposite sides are equal is a parallelogram.*



**Let C D represent a quadrilateral in which A C is equal to D B, and A D is equal to C B.**

*To prove that $A\,C\,B\,D$ is a parallelogram.*

*Suggestion 1.* Draw the diagonal $A\,B$.

*2.* Two definitions of a ▱ may now be used.

§§140 (c), 151.

*3.* How much of either is given in the theorem? What remains to be proved?

Work out a demonstration by use of each definition.

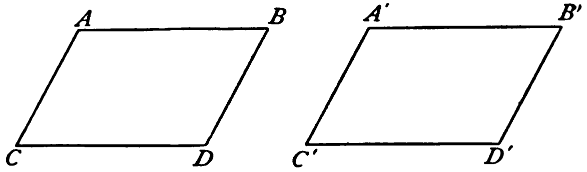
State the different definitions of a parallelogram that have now been established. Commit each to memory.

NOTE.—The pupil should carefully distinguish between a **property** of a figure and a **definition.**

A definition of a figure is such a description of it as serves to distinguish it from every other figure (§ 151). A property of a figure is a statement of it which is true, and may also be true of other figures (§ 148).

## PROPOSITION XLVI.

**153. Theorem.** *Two parallelograms which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal in all respects.*



**Let AD and A′D′ represent two parallelograms in which AB is equal to A′B′, AC is equal to A′C′ and angle A is equal to angle A′.**

*To prove that A D and A′ D′ are equal in all respects.*

*Suggestion 1.* Place ▱ *A D* upon ▱ *A′ D′* so that *A B* coincides with *A′ B′*, *A* upon *A′*, and *B* upon *B′*, and so that the two figures fall upon the same side of *A′ B′*.

2. What direction does *A C* take? Why?

3. Where does the point *C* fall? Why?

4. What direction does *C D* take? Why? *Ax.* 18.

5. Where does the point *D* fall? Why?

Complete the demonstration.

*Therefore* —

Ex. 76. If a parallelogram has one right angle it is a rectangle.

Ex. 77. A quadrilateral is a parallelogram if its diagonals bisect each other.

Ex. 78. A quadrilateral is a parallelogram if its opposite angles are equal.

### PROPOSITION XLVII.

154. **Theorem.** *The diagonals of a square or rhombus bisect the angles of the square or rhombus.*

**Let A B C D represent a square or rhombus, A C and B D the diagonals.**

*To prove that angles A, B, C,* and *D* are bisected.

*Suggestion.* Try to demonstrat this proposition by constructing the figure in your mind, and thus establish the theorem without the aid of a drawing.

---

Ex. 79. The diagonals of a square and rhombus intersect at right angles.

Try to find at least three methods by which to prove this exercise.

Ex. 80. Given a square *A D B C*. Draw diagonal *C D*.

Lay side *C B* upon the diagonal, *C* upon *C*, and *B* upon some point, as *E*. At *E* erect a perpendicular and extend to the side *D B*, meeting it at *F*.

Prove that *D E* is equal to *E F* and equal to *F B*.

*Suggestion.* Connect *C* and *F*.

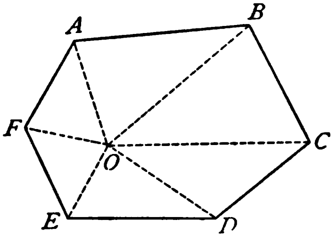
Ex. 81. If the diagonals of a parallelogram intersect at right angles, the figure is a square or rhombus.

Try to prove by at least two methods.

Ex. 82. If a straight line intercepted between parallel lines is bisected, any other straight line drawn through the point of bisection to meet the parallel lines is also bisected at that point. The two intersecting lines intercept equal portions on the parallel lines.

## PROPOSITION XLVIII.

**155. Theorem.** *The sum of the interior angles of any convex polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.*



**Let A D represent any convex polygon.**

*To prove that the sum of the interior angles of the polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.*

*Suggestion 1.* Connect each vertex with $O$, any point within the polygon.

*2.* If the polygon has $n$ sides, how many $\triangle$s are formed?

*3.* How many right $\angle$s in the sum of the interior angles of all of the $\triangle$s?

*4.* What is the sum of the $\angle$s about $O$?

*5.* Then, how many right $\angle$s in the sum of the interior $\angle$s of the polygon?

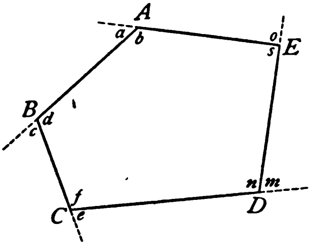
*Therefore —*

---

Ex. 83. Draw all possible diagonals of a polygon from a given vertex, and prove Prop. *XLVIII* by another method.

## PROPOSITION XLIX.

**156. Theorem.** *The sum of the exterior angles of any convex polygon, formed by extending each side through one vertex in order, is equal to four right angles.*



**Let A E D, etc., represent a convex polygon.**

*To prove that the sum of the exterior angles $a$, $c$, $e$, etc., is equal to four right angles.*

*Suggestion 1.* What is the sum of $\angle$s $a$ and $b$? Of $\angle$s $c$ and $d$? Of $\angle$s $f$ and $e$, etc.?

*2.* If the polygon has $n$ sides, the sum of all the exterior and interior $\angle$s equals how many right $\angle$s?

*3.* How many right $\angle$s in $\angle a + \angle c + \angle e$, etc. *Therefore* —

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Ex. 84. How many right angles in the sum of the interior angles of a pentagon? of a triangle? of a hexagon? of an octagon? of a decagon, etc.?

Ex. 85. How many right angles in the interior angles of a concave polygon? Compare with § 155.

## Review.

**157.** Name all the authorities:

1. (*a*) by which two triangles may be determined equal.

(*b*) by which two angles may be determined equal.

(*c*) by which two lines may be determined parallel?

(*d*) by which a line may be known to be perpendicular to another line.

(*e*) by which a quadrilateral may be known to be a parallelogram? a parallelogram to be a rectangle? a parallelogram to be a square or rhombus?

2. Name all the loci thus far demonstrated.

3. How may triangles be known to be isosceles?

4. State the relation that exists between angles of a triangle and sides opposite; between sides and angles opposite.

Ex. 86. What is the locus of a point two inches from a given line indefinite in length? Three inches from the line?

Ex. 87. Find a point $x$ that is four inches from a given line and also in another given line.

Use locus in determining $x$. Is it possible to find more than one $x$? Suppose the two lines are parallel and four inches apart; parallel and any other distance apart?

Ex. 88. Find point $x$ if it is in a given line and also equally distant from two intersecting lines. Ex. 72.

Ex. 89. Find point $x$ if it is equally distant from two intersecting lines and also a given distance from a given line. Ex. 86.

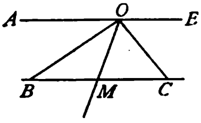
Ex. 90. Find point $x$ if it is equally distant from two intersecting lines, and also a given distance from a given line.

Ex. 91. Find point $x$ if it is equally distant from the extremities of a given line, and also a given distance from a given line.

Ex. 92. The three perpendicular bisectors of the sides of a triangle meet at a point.

*Sug.:* Use locus in demonstrating Exs. 90, 91, and 92.

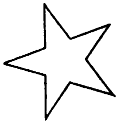
Ex. 93. In the figure, $AE$ and $BC$ are parallel, and $MO$ is a transversal. $BO$ bisects the angle $AOM$, and $CO$ bisects the angle $EOM$. Prove that $BM$ is equal to $MC$.



Ex. 94. If the middle points of the sides of a quadrilateral be joined in order, the figure formed is a parallelogram. *Sug.:* Draw the diagonals of the quadrilateral.

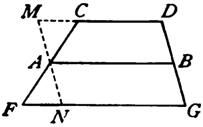
Ex. 95. The diameters of any quadrilateral bisect each other. Ex. 94.

Ex. 96. The sum of the angles at the vertices of a five-pointed star is equal to two right angles.



Ex. 97. If a line be drawn in a trapezoid, bisecting one of the legs and parallel to the bases, prove that it bisects the other leg also. *Sug.:* Draw a diagonal of the trapezoid.

Ex. 98. The sum of the bases of a trapezoid is equal to twice the diameter drawn to its legs.



Ex. 99. The difference between any two sides of a triangle is less than the third side.

# CHAPTER II.

## THE CIRCLE.

### Definitions.

158. A **circle** is a portion of a plane bounded by a curved line, all points of which are equally distant from a fixed point within.

159. The fixed point is the **center** of the circle, and the bounding line is the **circumference** of the circle.

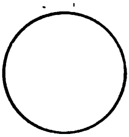


FIG. 1.

NOTE—In higher branches of mathematics the word circle is also used to denote what is here defined as the circumference; that is, the curved *line* bounding a portion of a plane instead of that portion of a plane itself, but in this book the above definitions will be adhered to.

160. A **radius** is any straight line drawn from the center of a circle to its circumference, as *A O*, Fig. 2.

161. A **diameter** is any straight line drawn through the center, terminated by the circumference, as *C B*, Fig. 2.

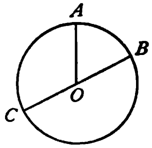


FIG. 2.

162. COROLLARY.—*From the definition of a circle, all radii of the same circle are equal; also, all diameters of the same circle are equal, each diameter being twice the radius.*

163. An **arc** of a circle is any portion of its circumference, as line *A M B*, Fig. 3.

164. A **chord** of a circle is a straight line joining any two points of the circumference, as *A B*, Fig. 3.

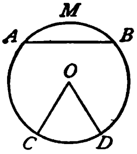


FIG. 3.

(*a*) When the extremities of a chord and arc are the same, the chord **subtends** the arc.

(*b*) In Fig. 3, the chord *A B* subtends the arc *A M B*, and also the arc *A C D B*. Thus, any chord subtends two arcs, which, together, make up the whole circumference. When an arc and its chord are spoken of, the smaller of the two arcs is always understood, unless the other is specifically stated. The terms **major** and **minor arcs** are sometimes used.

**165.** A **secant** of a circle is any straight line meeting the circumference in two points, and passing through the circumference in at least one of them, as *C D E*, Fig. 4.

A secant is a chord extended.

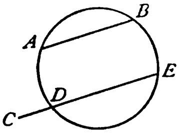


FIG. 4.

**166.** A **tangent** to a circle is a line which touches the circumference at one, and only one, point.

In Fig. 5, the line *M N* is tangent to the circle, and the circle is tangent to the line.

(*a*) The line may be straight or curved. If the curved line is a circumference, a circle is tangent to another circle.

(*b*) The point common to the line and the circle or to the two circles is the **point of contact** or **point of tangency.**

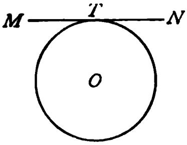


FIG. 5.

**167.** A **segment** of a circle is a portion of a circle bounded by an arc and its subtending chord, as segment *A M B*, Fig. 2.

**168.** A **sector** of a circle is a portion of a circle bounded by two radii and the intercepted arc. In Fig. 3, the sector *C O D* is bounded by the arc *C D*, and the radii *O C* and *O D*.

**169.** The arc intercepted between two radii **subtends** the angle made by the radii; arc $C D$, Fig. 3, subtends angle $C O D$.

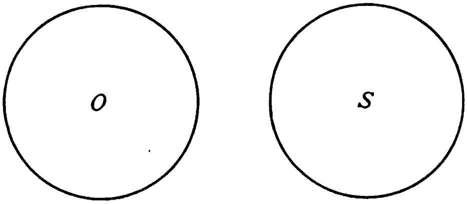
**170.** An **angle at the center** of a circle is an angle formed by two radii.

**171. Concentric circles** are circles that have the same center.

**172.** A circle may be **read** by naming the letter at the center of the circle, or by naming two or more letters on the circumference.

## PROPOSITION I.

**173. Theorem.** *Two circles are equal if the radius of one equals the radius of the other.*



**Let O and S represent two circles having equal radii.**

*To prove that circle O and circle S are equal.*

*Suggestion* 1. Place $\odot O$ upon $\odot S$, with the center $O$ upon the center $S$.

2. Where must the circumference of $O$ fall with respect to the circumference of $S$? Why?

*Therefore* —

### INDIRECT METHOD.

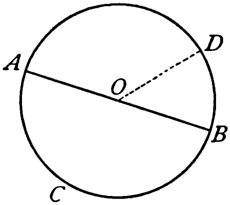
*Suggestion 1.* Same as above.

*2.* Suppose some part of the circumference of one ⊙ should fall outside of the circumference of the other ⊙, how would the radii compare?

*Therefore—*

### PROPOSITION II.

**174. Theorem.** *A diameter divides a circle into two equal parts.*



**Let A D B C represent a circle, and A B a diameter.**

*To prove that A B divides the circle into two equal parts.*

*Suggestion 1.* Let *D* be any point in the arc *A D B*, except *A* and *B*. Connect the center *O* with *D*.

*2.* Revolve the segment *A D B* upon the line *A O B*, as an axis, into the plane *A C B*.

*3.* Where does the point *D* fall? § 158.

*4.* Since *D* is any point in the arc *A D B*, compare the segments *A D B* and *A C B*. *Ax.* 13.

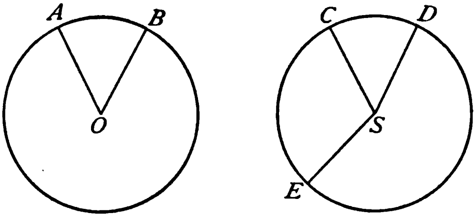
*Therefore—*

**175.** COROLLARY.—*A diameter divides the circumference into two equal parts.*

**176.** A **semicircle** is a segment of a circle bounded by a diameter and the arc it subtends, as *A D B*, Fig., § 174.

### PROPOSITION III.

**177. Theorem.** *In the same circle, or in equal circles, equal angles at the center intercept equal arcs at the circumference, and of two unequal angles, the greater intercepts the greater arc.*



**Let O and S represent equal circles, and let A O B and C S D represent equal angles at the centers O and S respectively.**

CASE I. *To prove that arc A B is equal to arc C D.*

*Suggestion 1.* Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *A O* upon *C S*. Where does the circumference of ⊙ *O* fall? Why?
Where does point *A* fall? Why?

*2.* What direction does *O B* take? Why?

*3.* What is the location of arc *A B*? Why?

*Therefore* —

**Let angle C S E be greater than angle A O B.**

CASE II. *To prove that arc E C is greater than arc A B.*

*Suggestion 1.* Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *O B* upon *S C*. Where does *O A* fall with respect to ∠ *C S E*? Why? § 17 (e).

2. Where does point $A$ fall with respect to arc $CE$? Why? Compare arc $AB$ and $CE$.

*Therefore —*

### PROPOSITION IV.

**178. Theorem.** CONVERSELY. *In the same circle, or in equal circles, equal arcs subtend equal angles at the center, and of two unequal arcs, the greater arc subtends the greater angle at the center.*

**Let O and S (§ 177) represent equal circles, and A B and C D equal arcs.**

CASE I. *To prove that angle $O$ is equal to angle $CSD$.*

*Suggestion.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, so that arcs $AB$ and $CD$ coincide. Why is this possible?

Complete the demonstration.

*Therefore —*

**Let arc E C be greater than arc A B.**

CASE II. *To prove angle $CSE$ is greater than angle $O$.*

*Suggestion 1.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, and $B$ upon $C$. Where does point $A$ fall with respect to arc $CE$? Why?

2. Where does line $OA$ fall? Why?

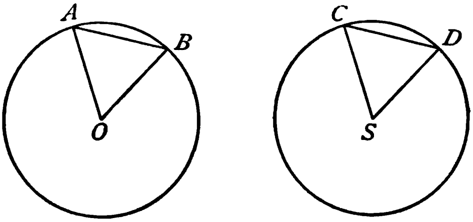
*Therefore —*

---

Ex. 100. The locus of a point at a given distance from a given point is the circumference of a circle, drawn with the given point as a center and with the length of the given distance as a radius. § 136.

### PROPOSITION V.

179. **Theorem.** *In the same circle, or in equal circles, chords which subtend equal arcs are equal.*



**Let O and S represent equal circles, in which arc A D is equal to arc C D.**

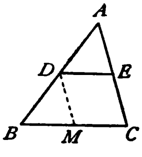
*To prove that chord A B is equal to chord C D.*

*Suggestion 1.* Draw the radii $O A$, $O B$, $S C$, and $S D$.
*2.* Compare $\angle$s $O$ and $S$. Give auth.
*3.* Compare $\triangle$s $A O B$ and $C S D$. Give auth.
*4.* Compare chord $A B$ with chord $C D$.
*Therefore* —

Ex. 101. If a line, parallel to the base of a triangle, bisects one side, it bisects the other side also, and is equal to one-half of the base. $A B$ is bisected at D, and $D E$ is parallel to $B C$.

Prove $A E = E C$, and $D E = \frac{1}{2} B C$.

*Suggestion.* Draw $D M \parallel$ to $A C$.

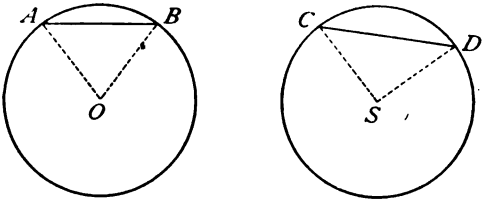


Ex. 102. With the same construction as in exercise 101, connect $E$ and $M$, and prove the exercise.

Ex. 103. If a line bisects the two legs of a triangle, prove that it is parallel to the base, and equals one-half of the base. Direct and indirect methods.

## PROPOSITION VI.

**180. Theorem.** *In the same circle, or in equal circles, two chords which subtend unequal arcs are unequal, that chord being greater which subtends the greater arc.*



**Let O and S represent equal circles, and let the arc A B be less than the arc C D.**

*To prove that the chord A B is less than the chord C D.*

*Suggestion 1.* Connect the extremities of the arcs with their respective centers.

*2.* Compare ∠ *S* with ∠ *O*. § 178.

*3.* Compare chord *C D* with chord *A B*. § 138.

*Therefore* —

---

Ex. 104. A diameter is greater than any other chord.

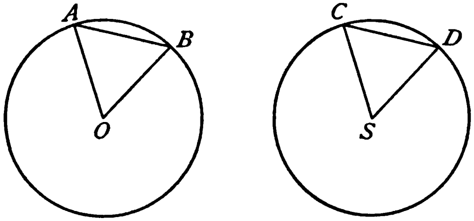
*Suggestion.* Draw any chord, not a diameter, and draw radii to the extremities. Compare the chord with the sum of the radii.

Ex. 105. If the diagonals of a parallelogram are equal, prove that the parallelogram is a rectangle.

Ex. 106. Given two parallel lines and a transversal. If each of the two interior angles on the same side of the transversal are bisected, prove that the bisectors are perpendicular to each other.

### PROPOSITION VII.

**181. Theorem.** CONVERSE OF PROPOSITION V. *In the same circle, or in equal circles, arcs which are subtended by equal chords are equal.*



**Let O and S represent equal circles in which the chord A B equals the chord C D.**

*To prove that the arc A B is equal to the arc C D.*

*Suggestion 1.* Draw radii $O A$, $O B$, $S C$, and $S D$.

*2.* Compare △s $A O B$ and $C S D$. Give auth.

*3.* Compare ∠s $O$ and $S$. Give auth.

*4.* Complete the demonstration.

*Therefore* —

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182. A **diameter** of a quadrilateral is a line which joins the middle points of two opposite sides.

Ex. 107. The diameter of a parallelogram divides it into two equal parallelograms.

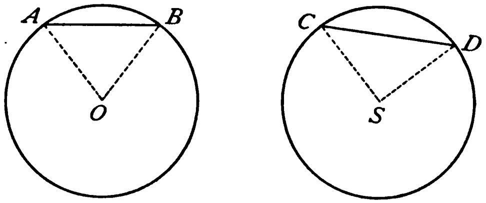
Ex. 108. The two diameters of a parallelogram bisect each other.

Ex. 109. If two adjacent sides of a rectangle are equal the figure is a square.

Ex. 110. Two lines perpendicular, respectively, to each of two intersecting lines, cannot be parallel.

## PROPOSITION VIII.

**183. Theorem.** CONVERSE OF PROPOSITION VI. *In the same circle, or in equal circles, two arcs which are subtended by unequal chords are unequal, and that arc is the greater which is subtended by the greater chord.*



**Let O and S represent equal circles, and let the chord A B be less than the chord C D.**

*To prove that the arc A B is less than the arc C D.*

*Suggestion 1.* Connect the extremities of the chords with the respective centers. Compare ∠ $O$ with ∠ $S$ in △s $A O B$ and $C S D$. § 139.

*2.* Compare arc $A B$ with arc $C D$. Give auth.

*Therefore* —

---

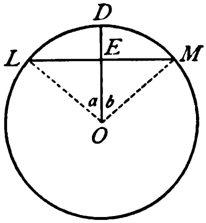
Ex. 111. If a circumference is divided into four equal parts and chords are drawn connecting their extremities, the resulting quadrilateral is a square.

Ex. 112. If one of the equal sides of an isosceles triangle is extended through the vertex, and the exterior angle formed is bisected, the bisector is parallel to the base.

Ex. 113. In § 153 draw the diagonals $C B$ and $C' B'$, and prove the proposition by comparison of triangles.

### PROPOSITION IX.

184. **Theorem.** *A radius which is perpendicular to a chord of a circle, bisects the chord and its subtended arc.*



**Let O D represent a radius perpendicular to the chord L M.**

*To prove that O D bisects the chord L M and arc L M.*

*Suggestion 1.* Let *E* be the point of intersection of *O D* and *L M*.

2. Compare △s *L E O* and *M E O*.

Complete the demonstration. § 177.

*Therefore —*

### PROPOSITION X.

185. **Theorem.** *A radius which bisects a chord of a circle is perpendicular to the chord.*

**Let the radius O D, in figure § 184, bisect the chord L M.**

*To prove O D is perpendicular to L M.*

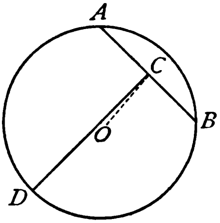
QUERY: How could this proposition be used to bisect an arc?

---

Ex. 114. A line which bisects a chord and its subtended arc cuts the center of the circle.

### PROPOSITION XI.

**186. Theorem.** *A line perpendicular to a chord at its middle point, passes through the center of the circle.*



**Let A B represent a chord, and C D a perpendicular to A B at its middle point C.**

*To prove that C D passes through the center of the circle.*

*Suggestion 1.* From *O*, the center of the circle, drop a ⊥ to the chord *A B*. § 184.

2. Where does *C O* lie with respect to *C D?* Why?

ANOTHER METHOD. *Suggestion 1.* What is the locus of a point equally distant from *A* and *B?*

2. Locate the center *O* with respect to this locus. Give auth.

*Therefore —*

187. COROLLARY.—*A line perpendicular to a chord at its middle point, bisects the subtended arc.*

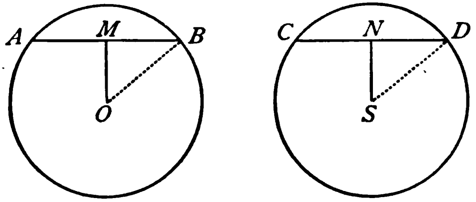
---

Ex. 115. If the line joining the middle points of two chords passes through the center of the circle, the chords are parallel.

Ex. 116. A radius drawn to the middle point of an arc bisects its subtending chord.

## PROPOSITION XII.

**188. Theorem.** *In the same circles or in equal circles, equal chords are equally distant from the center, and of two unequal chords, the greater is nearer the center.*



**Let AB and CD represent equal chords in the equal circles O and S, and OM and SN their respective distances from the centers O and S.**

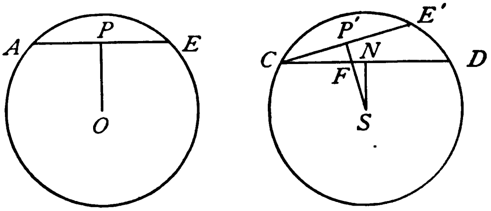
CASE I. *To prove that* $OM$ *is equal to* $SM$.

*Suggestion 1.* Draw the radii $OB$ and $SD$.

What relation must $MO$ and $NS$ bear to $AB$ and $CD$, respectively? § 130.

*2.* Compare $\triangle OMB$ with $\triangle SND$.

*3.* Compare $OM$ and $SN$.



**Let chord AE be less than chord CD, and let OP and SN represent the distances of the chords from their centers.**

Case II. *To prove that O P is greater than N S.*

*Suggestion 1.* Compare arc $A E$ and $C D$.

*2.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, and point $A$ upon point $C$.

Where does point $E$ fall with respect to point $D$? Sug. 1.

Where does line $A E$ fall with respect to segment $C E' D$?

*3.* Compare $N S$ and $F S$; $N S$ and $P' S$; $N S$ and $O P$.

*Therefore* —

Query: Why does $P' S$ cross the line $C D$? Sug. 2.

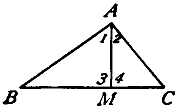
Corollary.—*In the same circle or in equal circles, two chords equally distant from the center are equal, and of two chords unequally distant from the center, that is the greater which is nearer the center.*

---

Ex. 117. Through a given point within a circle, the smallest possible chord is the one that is perpendicular to the radius passing through the point.

Ex. 118. $A B C$ is an isosceles triangle whose base is $B C$ and whose vertex is $A$. Extend $B A$ through $A$ to $O$, making $A O$ equal to $B A$. Connect $O$ and $C$. Prove that $O C$ is perpendicular to $B C$. (Ex. 112.)

Ex. 119. If a perpendicular is drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, prove that the two triangles formed are mutually equiangular.

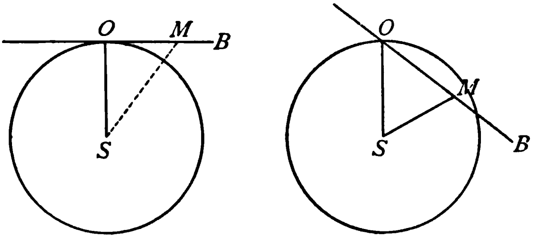


Ex. 120. Prove in Ex. 119 that each segment of the triangle is mutually equiangular to the whole triangle.

Ex. 121. What is the locus of a point equally distant from two parallel straight lines.

### Proposition XIII.

**189. Theorem.** *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*



**Let O B represent a straight line perpendicular to the radius O S at its extremity O.**

*To prove that O B is tangent to the circle.*

*Suggestion 1.* What must be proved to know that $O B$ is a tangent? § 166.

2 How much of what must be known is given in the theorem?

3. What remains to be proved?

4. Let $M$ represent any point except $O$ in line $O B$. Draw $S M$.

5. Compare $O S$ and $S M$ in respect to length.

6. Where is point $M$ in respect to the circle?

*Therefore* —

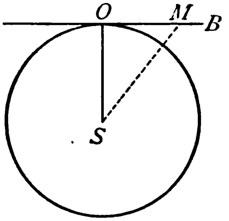
Query: How could this proposition be used to draw a tangent to a circle?

---

Ex. 122. The bisectors of the interior angles of a parallelogram form a rectangle. Suppose the parallelogram is equilateral?

## PROPOSITION XIV.

**190. Theorem.** CONVERSE OF PROPOSITION XIII. *If a straight line is tangent to a circle, the radius meeting it at the point of tangency is perpendicular to it.*



**Let OB represent a tangent to the circle S, O the point of tangency, and SO the radius drawn to the point of tangency.**

*To prove that S O is perpendicular to O B.*

*Suggestion 1.* Draw *S M*, a line from *S* to any point except *O* in *O B*.

*2.* Compare *S O* and *S M* in respect to length.

*3.* § 130.

*Therefore —*

**191.** COROLLARY 1.—*At any point in a circumference, one, and only one, tangent can be drawn.*

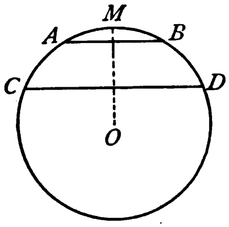
**192.** COROLLARY 2.—*A straight line perpendicular to a tangent at the point of tangency cuts the center of the circle.*

Ex. 123. All chords of a circle that are tangent to a concentric circle are equal.

Ex. 124. If two tangents are drawn to a circle at the ends of a diameter, they are parallel.

## PROPOSITION XV.

**193. Theorem.** *Arcs of a circle intercepted by parallel lines are equal.*



**Let A B and C D represent two parallel lines intercepting the arcs A C and B D.**

*To prove that arc A C is equal to arc B D.*

*Suggestion 1.* Drop a ⊥ from $O$ to $C D$, and extend it to meet the circumference, as at $M$.

*2.* How is $O M$ related to $A B$?

*3.* Complete the demonstration. § 184.

*Therefore* —

### Review.

**194.** State all the possible cases in which two parallel lines may intercept two arcs, and prove them equal in each case.

---

Ex. 125. Two tangents drawn to a circle from the same point are equal. *Suggestion.* Connect the center with the given point. § 190.

Ex. 126. If a tangent and a chord are parallel, prove that they intercept equal arcs.